

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 37 (1991)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** AFFINELY REGULAR INTEGRAL SIMPLICES  
**Autor:** Bacher, Roland  
**DOI:** <https://doi.org/10.5169/seals-58731>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

**Download PDF:** 14.03.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## AFFINELY REGULAR INTEGRAL SIMPLICES

by Roland BACHER

### 0. INTRODUCTION

We will consider the standard lattice  $\mathbf{Z}^n$  of the real vector space  $\mathbf{R}^n$  with  $n \geq 2$ . An *integral simplex* is a non-degenerate simplex of  $\mathbf{R}^n$  with all vertices in  $\mathbf{Z}^n$ . In this note, all simplices will be integral.

We will denote by  $\text{Aff}(\mathbf{Z}^n)$  the group of affine bijections of  $\mathbf{R}^n$  which preserve  $\mathbf{Z}^n$ ; it is the usual semi-direct product  $\mathbf{Z}^n \rtimes GL_n(\mathbf{Z})$ . The affine group  $\text{Aff}(\mathbf{Z}^n)$  acts naturally on the set of integral simplices in  $\mathbf{Z}^n$ .

For each integral simplex  $S$  we define

$$\text{Stab}(S) = \{g \in \text{Aff}(\mathbf{Z}^n) \mid g(S) = S\}$$

which is of course a subgroup of the group  $\sigma_{n+1}$  (group of permutations of  $n + 1$  objects), since there exists an injection in the group of permutations of the vertices of  $S$ .

*Definition 0.1.* A simplex  $S$  is called *affinely regular* if  $\text{Stab}(S)$  is equal to the whole group  $\sigma_{n+1}$ .

The definition of an affinely regular simplex is independent of the metric. For a discussion of integral simplices which are metrically regular one can consult [1] or [2] of the bibliography.

Two simplices  $S$  and  $S'$  are *equivalent* if there exists  $g \in \text{Aff}(\mathbf{Z}^n)$  such that  $g(S) = S'$ . The scope of this note is to find all equivalence classes of affinely regular simplices.

Let  $S$  be a simplex. Let us denote by  $\lambda S$  the image of the simplex  $S$  multiplied by some non-zero integer  $\lambda$ .

**PROPOSITION 0.2.** *The groups  $\text{Stab}(S)$  and  $\text{Stab}(\lambda S)$  are isomorphic for any integer  $\lambda \neq 0$ .*

*Proof.* Denote by  $\delta(\lambda)$  the linear automorphism  $x \mapsto \lambda x$  of  $\mathbf{R}^n$ . Let  $\phi_\lambda$  denote the endomorphism  $g \mapsto \delta(\lambda)g\delta(\lambda^{-1})$  of  $\text{Aff}(\mathbf{Z}^n)$ ; observe that  $\phi_\lambda$  is

one-to-one, but is not onto if  $|\lambda| \geq 2$ . Indeed, an affine bijection  $g \in \text{Aff}(\mathbf{Z}^n)$  is in the image of  $\phi_\lambda$  if and only if  $g$  preserves the sublattice  $\lambda\mathbf{Z}^n$  of  $\mathbf{Z}^n$ .

If  $g \in \text{Stab}(S)$ , then  $\phi_\lambda(g) \in \text{Stab}(\lambda S)$ . Consequently  $\phi_\lambda$  restricts to an injective homomorphism  $\psi_\lambda: \text{Stab}(S) \rightarrow \text{Stab}(\lambda S)$ . Let now  $h \in \text{Stab}(\lambda S)$ . We can write  $h = at$ , where  $a$  is in  $GL_n(\mathbf{Z})$  and where  $t$  is a translation. As  $a^{-1}$  preserves  $\lambda\mathbf{Z}^n$  (as any element of  $GL_n(\mathbf{Z})$  does), and as  $h$  preserves  $\lambda S$  one has

$$t(\lambda S) = a^{-1}h(\lambda S) = a^{-1}(\lambda S) \subset a^{-1}\lambda\mathbf{Z}^n$$

so that  $t$  preserves  $\lambda\mathbf{Z}^n$ . Hence  $h = at$  preserves  $\lambda\mathbf{Z}^n$ , so that  $h$  is in the image of  $\phi_\lambda$ . It follows that  $\psi_\lambda$  is an isomorphism onto.  $\square$

Caution: We have in fact proved that  $\text{Stab}(S)$  and  $\text{Stab}(\lambda S)$  are conjugate in  $\text{Aff}(\mathbf{Q}^n)$  but they are in general not conjugate in  $\text{Aff}(\mathbf{Z}^n)$ . This can be seen for instance by the fact that  $\text{Stab}(S)$  fixes the barycenter  $P$  of  $S$  and  $\text{Stab}(\lambda S)$  fixes  $\lambda P$ . But  $P$  and  $\lambda P$  are not necessarily in the same orbit of  $\text{Aff}(\mathbf{Z}^n)$ .

So  $\lambda S$  is affinely regular if and only if  $S$  is affinely regular. Hence we will be interested in minimal simplices.

*Definition 0.3.* An integral simplex  $S$  is *minimal* if, for every integral simplex  $T$  and for every integer  $\lambda \geq 1$  such that  $S$  is equivalent to  $\lambda T$ , we have  $\lambda = 1$ .

PROPOSITION 0.4. *Let  $S$  be an integral simplex of  $\mathbf{Z}^n$ . The following assertions are equivalent:*

- i)  $S$  is minimal.
- ii) For every integer  $\lambda \geq 2$  there exists no class of  $\mathbf{Z}^n$  modulo  $\lambda\mathbf{Z}^n$  which contains all the vertices of  $S$  modulo  $\lambda\mathbf{Z}^n$ .

*Proof.* Not (ii)  $\Rightarrow$  not (i). Let  $S$  be a simplex with all vertices in the same class of  $\mathbf{Z}^n$  modulo  $\lambda\mathbf{Z}^n$ . Let  $v_0$  be one of the vertices. The translate of  $S$  by  $-v_0$  is then a simplex with the coordinates of all vertices divisible by some  $\lambda \geq 2$ . This implies that  $S$  is not minimal.

Not (i)  $\Rightarrow$  not (ii). Let  $S$  be a non-minimal integral simplex. Hence there exists an integral simplex  $T$ , an integer  $\lambda \geq 2$ , an element  $g \in GL_n(\mathbf{Z})$  and a vector  $v \in \mathbf{Z}^n$  such that  $S = g(\lambda T) + v$ . But then all the vertices of  $S$  are in the class of  $v$  in  $\mathbf{Z}^n$  modulo  $\lambda\mathbf{Z}^n$ .  $\square$

The main subject of this note is to show the following theorem:

THEOREM 0.5. For  $n \geq 2$ , one has a bijection between the equivalence classes of minimal affinely regular integral simplices and the set of positive divisors of  $n + 1$  (including 1 and  $n + 1$ ). The bijection associates to the divisor  $k$  of  $n + 1$  the class of the simplex whose vertices are given by the columns of the following  $n \times n$  matrix

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & k-1 \\ 0 & 1 & 0 & \dots & 0 & k-1 \\ \cdot & \cdot & \cdot & \dots & \cdot & k-1 \\ 0 & 0 & 0 & \dots & 1 & k-1 \\ 0 & 0 & 0 & \dots & 0 & k \end{pmatrix}$$

and by the origin of  $\mathbf{Z}^n$ .

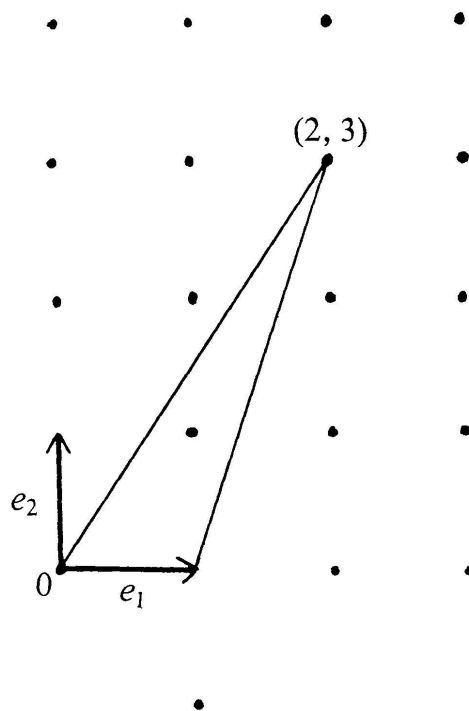
Proposition 0.4 implies that all representants in the theorem are minimal. Moreover, representants associated to distinct divisors  $k, k'$  of  $n + 1$  are non-equivalent since they are respectively of volumes  $k/n!$  and  $k'/n!$ .

The plan of the proof is as follows. We will introduce a family of particular simplices: those which have small faces. Then we dress the list of all small-faced affinely regular simplices (this gives us in fact the list of the theorem). Last, we prove that an affinely regular minimal simplex is necessarily small-faced.

Let us start with some examples:

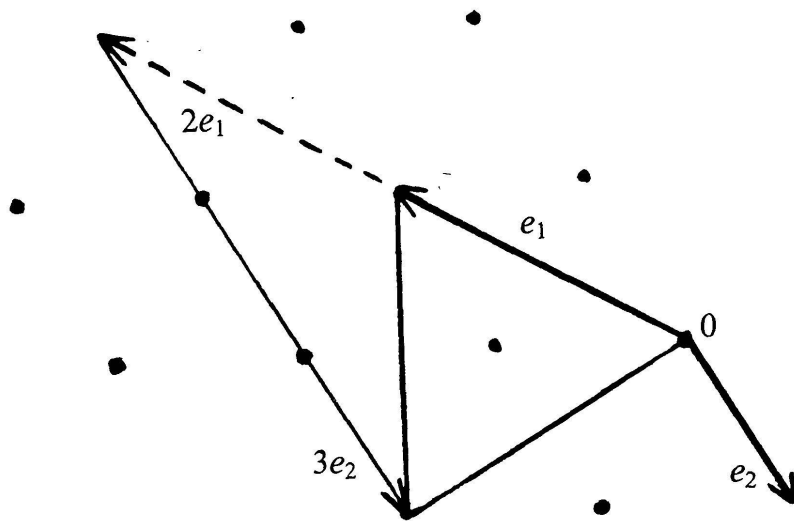
*Example 0.6.* Case where  $n = 2, k = 3$ .

In the standard lattice:





In the hexagonal lattice:



*Example 0.7.* Case where  $n = 3, k = 2$ .

Let  $C = [0, 1]^3$  be the standard cube of  $\mathbf{R}^3$ . Let  $\Delta$  be the tetrahedron defined by the vertices of the cube of which the sum of the coordinates is even. It is easy to see that  $\Delta$  is affinely regular and that the linear transformation defined by

$$e_1 \mapsto -e_3, \quad e_2 \mapsto e_1 + e_3, \quad e_3 \mapsto e_2 + e_3$$

(where  $(e_1, e_2, e_3)$  is the standard basis of  $\mathbf{R}^3$ ) sends  $\Delta$  to the representant given in Theorem 0.5.

## 1. SIMPLICES WITH SMALL FACES

*Definition 1.1.* An integral simplex  $S$  is said to have *small faces* if, for each hyperplan  $H$  containing a  $(n - 1)$ -face of  $S$ , the vertices of  $S$  contained in  $H$  constitute an affine  $\mathbf{Z}$ -basis of  $\mathbf{Z}^n \cap H$ .

A *numerotation* of an integral simplex  $S$  is an enumeration

$$v = (v_0, v_1, \dots, v_n)$$

of the vertices of  $S$ . We will denote by  $S_v$  the simplex  $S$  with numerotation  $v$ . The group  $\text{Aff}(\mathbf{Z}^n)$  acts naturally on the set of numerated simplices and we

will say that  $S_v$  (with  $v = (v_0, \dots, v_n)$ ) is equivalent to  $S_{v'}$  (with  $v' = (v'_0, \dots, v'_n)$ ) if there exists  $g \in \text{Aff}(\mathbf{Z}^n)$  such that  $g(v_i) = v'_i$  for all  $i$ .

The group  $\sigma_{n+1}$  acts on the set of numerated simplices: If  $s \in \sigma_{n+1}$  is a permutation of  $\{0, \dots, n\}$  and if  $S_v$  is an integral simplex numerated by  $v = (v_0, \dots, v_i, \dots, v_n)$ , we define

$$s \cdot S_v = S_{s \cdot v}$$

where

$$s \cdot v = (v_{s^{-1}(0)}, \dots, v_{s^{-1}(i)}, \dots, v_{s^{-1}(n)}) .$$

This action of  $\sigma_{n+1}$  commutes with that of  $\text{Aff}(\mathbf{Z}^n)$ , hence  $\sigma_{n+1}$  acts also on the equivalence classes of numerated simplices modulo  $\text{Aff}(\mathbf{Z}^n)$ .

An integral simplex  $S$  is affinely regular if and only if the stabilizer  $\text{Stab}(S)$  operates transitively on the numerations of  $S$ .

Let us recall an elementary and well-known lemma:

LEMMA 1.2. *Let  $v_1, \dots, v_{n-1}$  be linearly independent vectors of  $\mathbf{Z}^n$ . Let  $H$  be the hyperplane of  $\mathbf{R}^n$  generated by the  $v_i$ 's, and suppose that  $v_1, \dots, v_{n-1}$  form a basis of the sublattice  $H \cap \mathbf{Z}^n$ .*

*Then we can complete  $v_1, \dots, v_{n-1}$  to a basis of  $\mathbf{Z}^n$ .*

*Proof.* See for instance the Corollary in Bourbaki, *Algèbre*, chap. VII, §4, No. 3.

From this point until the end of section 2,  $k$  will be some fixed natural integer. Let now  $S_v$  be a numerated simplex with small faces, of volume  $k/n!$ . Let  $\{e_1, \dots, e_n\}$  be the canonical basis of  $\mathbf{Z}^n \subset \mathbf{R}^n$ . The lemma implies that there exists  $g \in \text{Aff}(\mathbf{Z}^n)$  such that  $gS_v$  has vertices

$$v_0 = 0, v_1 = e_1, \dots, v_{n-1} = e_{n-1}, v_n = ke_n + \sum_{i=1}^{n-1} a_i e_i$$

where the  $a_i$ 's are integers. An easy calculation shows that the  $a_i$ 's are well defined (mod  $k$ ).

Let us associate to  $S_v$  the element  $(\alpha_1, \dots, \alpha_{n-1})$ , where  $\alpha_i$  is the class of  $a_i$  (mod  $k$ ). This gives us a map  $\rho_k: \Sigma_k \rightarrow (\mathbf{Z}/k\mathbf{Z})^{n-1}$ , where  $\Sigma_k$  is the set of numerated simplices with small faces of volume  $k/n!$ . For  $S_v \in \Sigma_k$ , the element  $\rho_k(S_v)$  depends only of the equivalence class of  $S_v$  modulo  $\text{Aff}(\mathbf{Z}^n)$  and this allows us to define an action of  $\sigma_{n+1}$  on  $\rho_k(\Sigma_k)$ .

2. ACTION OF  $\sigma_{n+1}$  ON  $\rho_k(\Sigma_k)$ 

In this section, we show how the  $n$  usual generators  $(0, 1), \dots, (n-1, n)$  of the group  $\sigma_{n+1}$  act on the subset  $\rho_k(\Sigma_k)$  of  $(\mathbf{Z}/k\mathbf{Z})^{n-1}$ .

LEMMA 2.1. *Let  $(\alpha_1, \dots, \alpha_{n-1}) \in \rho_k(\Sigma_k)$ . Then*

$$\begin{aligned} (0, 1) \cdot (\alpha_1, \alpha_2, \dots, \alpha_i, \dots, \alpha_{n-1}) &= ((1 - \alpha_1 - \alpha_2 - \dots - \alpha_{n-1}), \alpha_2, \dots, \alpha_i, \dots, \alpha_{n-1}) \\ (i, i+1) \cdot (\alpha_1, \dots, \alpha_i, \alpha_{i+1}, \dots, \alpha_{n-1}) &= (\alpha_1, \dots, \alpha_{i+1}, \alpha_i, \dots, \alpha_{n-1}), \quad 1 \leq i \leq n-2 \\ (n-1, n) \cdot (\alpha_1, \dots, \alpha_i, \dots, \alpha_{n-2}, \alpha_{n-1}) &= (-\alpha_1 \alpha_{n-1}^{-1}, \dots, -\alpha_i \alpha_{n-1}^{-1}, \dots, -\alpha_{n-2} \alpha_{n-1}^{-1}, \alpha_{n-1}^{-1}) \end{aligned}$$

*In particular, if  $(\alpha_1, \dots, \alpha_{n-1}) \in \rho_k(\Sigma_k)$ , then  $\alpha_{n-1}$  is invertible (mod  $k$ ).*

*Proof.* Let us show the first equality. Let  $(\alpha_1, \dots, \alpha_{n-1}) \in \rho_k(\Sigma_k)$ . Consider the numerated simplex  $S_v$  with vertices

$$(2.2) \quad v_0 = 0, v_i = e_i, 1 \leq i \leq n-1, v_n = ke_n + \sum_{i=1}^{n-1} a_i e_i$$

( $a_i$  representant of the class  $\alpha_i$ ).

Let us identify  $\mathbf{R}^n$  with the points of the hyperplane  $H$  of  $\mathbf{R}^{n+1}$  defined by  $x_{n+1} = 1$ . By identifying the elements of  $\mathbf{R}^n$  with vector columns we see that

$$0 \mapsto \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{pmatrix}, \quad e_i \mapsto \begin{pmatrix} 0 \\ \cdot \\ 1 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{pmatrix}$$

where  $0$  is the origin in  $\mathbf{R}^n$  and  $\{e_1, \dots, e_n\}$  is the usual basis of  $\mathbf{R}^n$  and the  $1$  in the second matrix is where you think it should be, namely at the  $i$ -th row. There exists a natural injection of the group  $\text{Aff}(\mathbf{Z}^n)$  in the subgroup of  $GL_{n+1}(\mathbf{Z})$  which preserves  $H$ . It is easy to check that the following matrix  $M \in GL_{n+1}(\mathbf{Z})$  exchanges  $v_0$  and  $v_1$ , preserves  $v_i$  for  $2 \leq i < n$  and sends  $v_n$  to the element  $ke_n + (1 - \sum_{i=1}^{n-1} a_i)e_1 + \sum_{i=2}^{n-1} a_i e_i$ :

$$M = \begin{pmatrix} -1 & -1 & -1 & \dots & -1 & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}.$$

The calculations for the transposition  $(i, i+1)$  are immediate if  $1 \leq i \leq n-2$ .

Finally, let us consider the last equality: We take again the simplex  $S_v$  with vertices as in (2.2). Since  $S_v$  is small-faced, there exists a simplex  $S'_\mu$ ,  $(n-1)$  integers  $b_1, \dots, b_{n-1}$  and an element  $g \in \text{Aff}(\mathbf{Z}^n)$  such that the vertices of  $S'_\mu$  are

$$v'_0 = 0, v'_i = e_i, 1 \leq i \leq n-1, v'_n = ke_n + \sum_{i=1}^{n-1} b_i e_i$$

and such that  $g(v_i) = v'_i$  for  $0 \leq i \leq n-2$ ,  $g(v_{n-1}) = v'_n$  and  $g(v_n) = v'_{n-1}$ . Since  $v_0 = v'_0 = 0$  we have  $g(0) = 0$  and  $g$  is in fact in  $GL_n(\mathbf{Z})$ . The matrix of  $g$  with respect to the standard basis is

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & b_1 \\ 0 & 1 & 0 & \dots & 0 & 0 & b_2 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 & 0 & b_{n-2} \\ 0 & 0 & 0 & \dots & 0 & 1 & b_{n-1} \\ 0 & 0 & 0 & \dots & 0 & 0 & k \end{pmatrix} \begin{pmatrix} 1 & 0 & \dots & 0 & a_1 & 0 \\ 0 & 1 & \dots & 0 & a_2 & 0 \\ \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 1 & a_{n-2} & 0 \\ 0 & 0 & \dots & 0 & a_{n-1} & 1 \\ 0 & 0 & \dots & 0 & k & 0 \end{pmatrix}^{-1}$$

this gives

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & b_1 \\ 0 & 1 & 0 & \dots & 0 & 0 & b_2 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 & 0 & b_{n-2} \\ 0 & 0 & 0 & \dots & 0 & 1 & b_{n-1} \\ 0 & 0 & 0 & \dots & 0 & 0 & k \end{pmatrix} \frac{1}{k} \begin{pmatrix} k & 0 & 0 & \dots & 0 & 0 & -a_1 \\ 0 & k & 0 & \dots & 0 & 0 & -a_2 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & k & 0 & -a_{n-2} \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & k & -a_{n-1} \end{pmatrix}.$$

But since  $g \in GL_n(\mathbf{Z})$  this implies that

$$-a_i - b_i a_{n-1} \equiv 0 \pmod{k} \quad 1 \leq i \leq n-2.$$

Let us now suppose that  $a_{n-1}$  is not invertible  $(\text{mod } k)$ . Then there exists some prime  $p$  dividing both  $k$  and  $a_{n-1}$ . But then the prime  $p$  divides  $a_i$  too for every  $i$ . So  $p$  divides all coefficients of the vector  $v_n - v_0$  which is an edge of  $S_v$ . But then  $S_v$  is not small-faced which contradicts the fact that  $(\alpha_1, \dots, \alpha_{n-1}) \in \rho_k(\Sigma_k)$ . So we have proved that  $a_{n-1}$  is invertible  $(\text{mod } k)$ . And the  $b_i$ 's satisfy

$$b_i \equiv -a_i a_{n-1}^{-1} \pmod{k} \quad 1 \leq i \leq n-2 \quad \text{and} \quad b_{n-1} \equiv a_{n-1}^{-1} \pmod{k}.$$

This proves the last equality.  $\square$

## 3. LIST OF SMALL-FACED AFFINELY REGULAR SIMPLICES

An integral simplex  $S$  with small faces is affinely regular if and only if the numerated simplices  $S_v$  and  $S_{v'}$  are equivalent for each pair  $v, v'$  of enumerations of  $S$ . In other terms, an integral simplex  $S$  with small faces of volume  $k/n!$  is affinely regular if and only if  $\rho_k(S_v) = \rho_k(S_{v'})$  for all enumerations  $v$  and  $v'$ , hence if and only if the element  $\rho_k(S_v)$  is a fixed point under the action of  $\sigma_{n+1}$  on  $\rho_k(\Sigma_k)$ .

It is sufficient for  $\rho_k(S_v)$  to be fixed under the action of a set of generators in order to be a fixpoint of  $\sigma_{n+1}$  acting on  $\rho_k(\Sigma_k)$ . Let us suppose that  $(\alpha_1, \dots, \alpha_{n-1})$  is a fixpoint of  $\rho_k(\Sigma_k)$ . Then, for all  $i \in \{1, \dots, n-2\}$ :

$$\begin{aligned} (i, i+1) \cdot (\alpha_1, \dots, \alpha_i, \alpha_{i+1}, \dots, \alpha_{n-1}) &= (\alpha_1, \dots, \alpha_{i+1}, \alpha_i, \dots, \alpha_{n-1}) \\ &= (\alpha_1, \dots, \alpha_i, \alpha_{i+1}, \dots, \alpha_{n-1}) \end{aligned}$$

implies  $\alpha_i \equiv \alpha \pmod{k}$  for some  $\alpha \in \mathbf{Z}/k\mathbf{Z}$ .

Furthermore

$$(n-1, n) \cdot (\alpha, \dots, \alpha, \alpha) = (-\alpha\alpha^{-1}, \dots, -\alpha\alpha^{-1}, \alpha^{-1}) = (\alpha, \dots, \alpha)$$

gives  $\alpha \equiv -\alpha\alpha^{-1} \equiv -1 \pmod{k}$ .

Finally

$$(0, 1) \cdot (-1, -1, \dots, -1) = (1 - (n-1)(-1), -1, \dots, -1) = (-1, \dots, -1)$$

implies  $-1 \equiv 1 - (n-1)(-1) \pmod{k}$  namely  $0 \equiv n+1 \pmod{k}$  namely  $k \mid (n+1)$ .

This shows that the simplices listed in the theorem are exactly all the affinely regular simplices with small faces.

We have yet to show that any affinely regular minimal simplex is small-faced. This will be the aim of the next paragraph.

## 4. ANY AFFINELY REGULAR MINIMAL SIMPLEX HAS SMALL FACES

Lemma 1.2 implies the following corollary:

**COROLLARY 4.1.** *Every integral simplex of  $\mathbf{Z}^n$  with numerated vertices is equivalent to an integral simplex with vertex  $v_0$  at 0 and vertex  $v_i$  at the  $i$ -th vector-column of an upper triangular matrix ( $i > 0$ ).*

Regularity and Proposition 0.4 imply almost immediately the following:

*Remark 4.2.* Let  $S$  be an affinely regular simplex. Then  $S$  is minimal if and only if the interior of each edge of  $S$  is without integral points.

Let us start with the proof that each affinely regular minimal simplex is small-faced.

Consider an affinely regular minimal simplex  $S$  of  $\mathbf{Z}^2$ . Corollary 4.1 implies that  $S$  is equivalent to a simplex  $S'$  with vertices at 0 and at the vector-columns of a matrix of the type  $\begin{pmatrix} l & a \\ 0 & k \end{pmatrix}$ . Remark 4.2 implies that the integer  $l$  is equal to  $\pm 1$ .

By exchanging  $S'$  with an equivalent simplex if necessary, we can suppose that  $S'$  has its vertices at 0 and at the vector-columns of a matrix of type  $\begin{pmatrix} 1 & a \\ 0 & k \end{pmatrix}$  with  $k$  a positive integer. The affine regularity now implies that  $S'$  (and hence  $S$ ) is small-faced.

Hence the theorem holds for  $n = 2$ .

Induction:  $(n - 1) \Rightarrow (n)$ .

Let  $S_v$  be a numerated affinely regular minimal simplex of  $\mathbf{Z}^n$  with underlying simplex  $S$ . Corollary 4.1 implies that, after some suitable choice of an equivalent simplex, we can suppose that  $v_0 = 0$  and  $v_i$  is the  $i$ -th vector-column of an upper triangular  $n \times n$  matrix  $T$ .

The  $(n - 1)$ -face containing  $v_0, v_1, \dots, v_{n-1}$  is an affinely regular simplex of  $\mathbf{Z}^{n-1}$  and Remark 4.2 shows that it is minimal. So Lemma 1.2 and the induction hypothesis imply that, possibly after a suitable change of  $S_v$ , the matrix  $T$  is of the form

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & l-1 & a_1 \\ 0 & 1 & 0 & \dots & 0 & l-1 & a_2 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 & l-1 & a_{n-2} \\ 0 & 0 & 0 & \dots & 0 & l & a_{n-1} \\ 0 & 0 & 0 & \dots & 0 & 0 & k \end{pmatrix}$$

where  $l$  and  $k$  are positive integers, and where  $l$  divides  $n$  by induction hypothesis.

Set  $\mu = n/l \in \mathbf{N}$ .

The barycenter of  $\mu v_0, \mu v_1, \dots, \mu v_{n-1}$  is  $e_1 + e_2 + \dots + e_{n-1}$ .

Since  $S_v$  is affinely regular, there exists for all  $i \in \{0, 1, \dots, n\}$  an element  $g_i \in \text{Aff}(\mathbf{Z}^n)$  which sends

$$\mu v_0, \dots, \mu v_{n-1}, \widehat{\mu v_n} \quad \text{to} \quad \mu v_0, \dots, \widehat{\mu v_i}, \dots, \mu v_n;$$

the barycenter of  $\mu v_0, \dots, \widehat{\mu v_i}, \dots, \mu v_n$ , which is  $g_i(e_1 + e_2 + \dots + e_{n-1})$ , is consequently also in  $\mathbf{Z}^n$ .

So the barycenters of all faces of  $\mu S_\nu$  are in  $\mathbf{Z}^n$  and they are the vertices of an integral simplex  $S'$ .

Calculating the first coordinate of the barycenter of  $\widehat{\mu v_0}, \mu v_1, \dots, \mu v_n$  we see that  $n$  divides  $\mu + \mu(l-1) + \mu a_1$ .

Calculating the first coordinate of the barycenter of  $\mu v_0, \widehat{\mu v_1}, \mu v_2, \dots, \mu v_n$ , we see that  $n$  divides  $\mu(l-1) + \mu a_1$ .

So the integer  $n$  divides  $\mu$  too but this implies that  $\mu = n$  and hence  $l = 1$ . This and the affine regularity imply that  $S$  is small-faced.  $\square$

The notions of affine regularity and of integrality may both be generalized to other polytopes, such as hypercubes, cross-polytopes, hexagones in dimension 2 or exceptional polytopes in dimension 4. We plan to consider these in a further paper.

#### BIBLIOGRAPHY

- [1] EHRHART, E. Sur les polygones et les polyèdres entiers. *L'Enseignement Mathématique, T. V, Fascicule 2* (1959), 81-85.
- [2] MACDONALD, I. G. Regular simplexes with integral vertices. *C. R. Math. Rep. Acad. Sci. Canada, Vol. IX, No. 4*, August 1987, 189-193.
- [3] BOURBAKI, N. *Algèbre*, chapitres 4 à 7. Masson, Paris, 1981.

(Reçu le 20 novembre 1990)

Roland Bacher

Section de mathématiques  
2-4, rue du Lièvre  
CP 240  
CH-1211 Genève 24