

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 37 (1991)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: AFFINELY REGULAR INTEGRAL SIMPLICES
Autor: Bacher, Roland
Kapitel: 0. Introduction
DOI: <https://doi.org/10.5169/seals-58731>

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AFFINELY REGULAR INTEGRAL SIMPLICES

by Roland BACHER

0. INTRODUCTION

We will consider the standard lattice \mathbf{Z}^n of the real vector space \mathbf{R}^n with $n \geq 2$. An *integral simplex* is a non-degenerate simplex of \mathbf{R}^n with all vertices in \mathbf{Z}^n . In this note, all simplices will be integral.

We will denote by $\text{Aff}(\mathbf{Z}^n)$ the group of affine bijections of \mathbf{R}^n which preserve \mathbf{Z}^n ; it is the usual semi-direct product $\mathbf{Z}^n \rtimes GL_n(\mathbf{Z})$. The affine group $\text{Aff}(\mathbf{Z}^n)$ acts naturally on the set of integral simplices in \mathbf{Z}^n .

For each integral simplex S we define

$$\text{Stab}(S) = \{g \in \text{Aff}(\mathbf{Z}^n) \mid g(S) = S\}$$

which is of course a subgroup of the group σ_{n+1} (group of permutations of $n + 1$ objects), since there exists an injection in the group of permutations of the vertices of S .

Definition 0.1. A simplex S is called *affinely regular* if $\text{Stab}(S)$ is equal to the whole group σ_{n+1} .

The definition of an affinely regular simplex is independent of the metric. For a discussion of integral simplices which are metrically regular one can consult [1] or [2] of the bibliography.

Two simplices S and S' are *equivalent* if there exists $g \in \text{Aff}(\mathbf{Z}^n)$ such that $g(S) = S'$. The scope of this note is to find all equivalence classes of affinely regular simplices.

Let S be a simplex. Let us denote by λS the image of the simplex S multiplied by some non-zero integer λ .

PROPOSITION 0.2. *The groups $\text{Stab}(S)$ and $\text{Stab}(\lambda S)$ are isomorphic for any integer $\lambda \neq 0$.*

Proof. Denote by $\delta(\lambda)$ the linear automorphism $x \mapsto \lambda x$ of \mathbf{R}^n . Let ϕ_λ denote the endomorphism $g \mapsto \delta(\lambda)g\delta(\lambda^{-1})$ of $\text{Aff}(\mathbf{Z}^n)$; observe that ϕ_λ is

one-to-one, but is not onto if $|\lambda| \geq 2$. Indeed, an affine bijection $g \in \text{Aff}(\mathbf{Z}^n)$ is in the image of ϕ_λ if and only if g preserves the sublattice $\lambda\mathbf{Z}^n$ of \mathbf{Z}^n .

If $g \in \text{Stab}(S)$, then $\phi_\lambda(g) \in \text{Stab}(\lambda S)$. Consequently ϕ_λ restricts to an injective homomorphism $\psi_\lambda: \text{Stab}(S) \rightarrow \text{Stab}(\lambda S)$. Let now $h \in \text{Stab}(\lambda S)$. We can write $h = at$, where a is in $GL_n(\mathbf{Z})$ and where t is a translation. As a^{-1} preserves $\lambda\mathbf{Z}^n$ (as any element of $GL_n(\mathbf{Z})$ does), and as h preserves λS one has

$$t(\lambda S) = a^{-1}h(\lambda S) = a^{-1}(\lambda S) \subset a^{-1}\lambda\mathbf{Z}^n$$

so that t preserves $\lambda\mathbf{Z}^n$. Hence $h = at$ preserves $\lambda\mathbf{Z}^n$, so that h is in the image of ϕ_λ . It follows that ψ_λ is an isomorphism onto. \square

Caution: We have in fact proved that $\text{Stab}(S)$ and $\text{Stab}(\lambda S)$ are conjugate in $\text{Aff}(\mathbf{Q}^n)$ but they are in general not conjugate in $\text{Aff}(\mathbf{Z}^n)$. This can be seen for instance by the fact that $\text{Stab}(S)$ fixes the barycenter P of S and $\text{Stab}(\lambda S)$ fixes λP . But P and λP are not necessarily in the same orbit of $\text{Aff}(\mathbf{Z}^n)$.

So λS is affinely regular if and only if S is affinely regular. Hence we will be interested in minimal simplices.

Definition 0.3. An integral simplex S is *minimal* if, for every integral simplex T and for every integer $\lambda \geq 1$ such that S is equivalent to λT , we have $\lambda = 1$.

PROPOSITION 0.4. *Let S be an integral simplex of \mathbf{Z}^n . The following assertions are equivalent:*

- i) S is minimal.
- ii) For every integer $\lambda \geq 2$ there exists no class of \mathbf{Z}^n modulo $\lambda\mathbf{Z}^n$ which contains all the vertices of S modulo $\lambda\mathbf{Z}^n$.

Proof. Not (ii) \Rightarrow not (i). Let S be a simplex with all vertices in the same class of \mathbf{Z}^n modulo $\lambda\mathbf{Z}^n$. Let v_0 be one of the vertices. The translate of S by $-v_0$ is then a simplex with the coordinates of all vertices divisible by some $\lambda \geq 2$. This implies that S is not minimal.

Not (i) \Rightarrow not (ii). Let S be a non-minimal integral simplex. Hence there exists an integral simplex T , an integer $\lambda \geq 2$, an element $g \in GL_n(\mathbf{Z})$ and a vector $v \in \mathbf{Z}^n$ such that $S = g(\lambda T) + v$. But then all the vertices of S are in the class of v in \mathbf{Z}^n modulo $\lambda\mathbf{Z}^n$. \square

The main subject of this note is to show the following theorem:

THEOREM 0.5. For $n \geq 2$, one has a bijection between the equivalence classes of minimal affinely regular integral simplices and the set of positive divisors of $n + 1$ (including 1 and $n + 1$). The bijection associates to the divisor k of $n + 1$ the class of the simplex whose vertices are given by the columns of the following $n \times n$ matrix

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & k-1 \\ 0 & 1 & 0 & \dots & 0 & k-1 \\ \cdot & \cdot & \cdot & \dots & \cdot & k-1 \\ 0 & 0 & 0 & \dots & 1 & k-1 \\ 0 & 0 & 0 & \dots & 0 & k \end{pmatrix}$$

and by the origin of \mathbf{Z}^n .

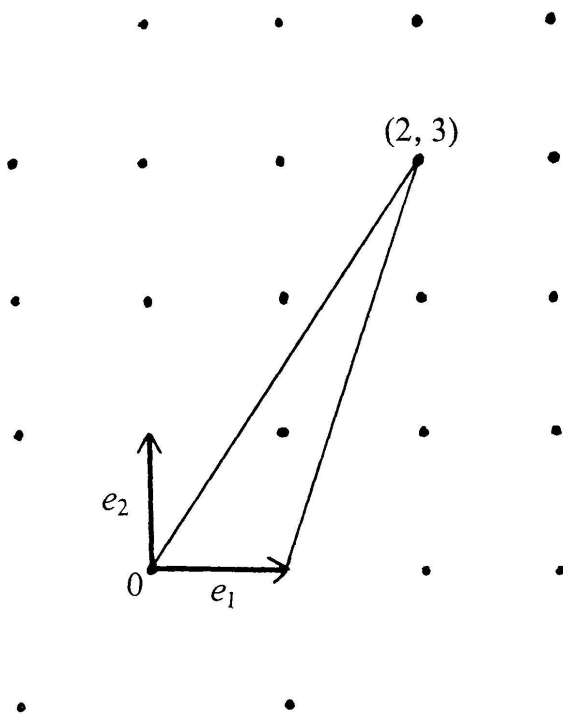
Proposition 0.4 implies that all representants in the theorem are minimal. Moreover, representants associated to distinct divisors k, k' of $n + 1$ are non-equivalent since they are respectively of volumes $k/n!$ and $k'/n!$.

The plan of the proof is as follows. We will introduce a family of particular simplices: those which have small faces. Then we dress the list of all small-faced affinely regular simplices (this gives us in fact the list of the theorem). Last, we prove that an affinely regular minimal simplex is necessarily small-faced.

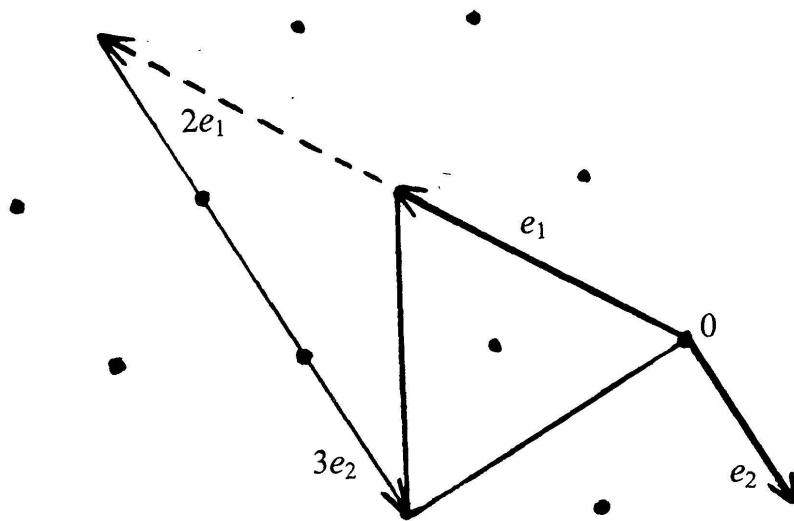
Let us start with some examples:

Example 0.6. Case where $n = 2, k = 3$.

In the standard lattice:



In the hexagonal lattice:



Example 0.7. Case where $n = 3, k = 2$.

Let $C = [0, 1]^3$ be the standard cube of \mathbf{R}^3 . Let Δ be the tetrahedron defined by the vertices of the cube of which the sum of the coordinates is even. It is easy to see that Δ is affinely regular and that the linear transformation defined by

$$e_1 \mapsto -e_3, \quad e_2 \mapsto e_1 + e_3, \quad e_3 \mapsto e_2 + e_3$$

(where (e_1, e_2, e_3) is the standard basis of \mathbf{R}^3) sends Δ to the representant given in Theorem 0.5.

1. SIMPLICES WITH SMALL FACES

Definition 1.1. An integral simplex S is said to have *small faces* if, for each hyperplan H containing a $(n - 1)$ -face of S , the vertices of S contained in H constitute an affine \mathbf{Z} -basis of $\mathbf{Z}^n \cap H$.

A *numerotation* of an integral simplex S is an enumeration

$$v = (v_0, v_1, \dots, v_n)$$

of the vertices of S . We will denote by S_v the simplex S with numerotation v . The group $\text{Aff}(\mathbf{Z}^n)$ acts naturally on the set of numerated simplices and we