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**Autor:** Davis, James F. / Livingston, Charles  
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$$\sigma = 1 + g + g^2 + \dots + g^{p-1}$$

$$\delta = 1 - g$$

be elements of the group ring  $\mathbf{F}_p[G]$ . Note that  $\delta\sigma = 0 = \sigma\delta$  and  $\delta^{p-1} = \sigma$ . We consider the following chain complexes of  $\mathbf{F}_p[t, t^{-1}]$ -modules (all homology is with  $\mathbf{F}_p$ -coefficients).

$$\begin{aligned} 0 &\rightarrow C_*(X^G) \rightarrow C_*(\bar{X}) \xrightarrow{\text{tr}} \sigma C_*(X) \rightarrow 0 \\ 0 &\rightarrow \delta C_*(X) \oplus C_*(X^G) \rightarrow C_*(X) \xrightarrow{\sigma} \sigma C_*(X) \rightarrow 0 \\ 0 &\rightarrow \sigma C_*(X) \rightarrow \delta C_*(X) \xrightarrow{\delta} \delta^2 C_*(X) \rightarrow 0 \\ &\quad \vdots \\ 0 &\rightarrow \sigma C_*(X) \rightarrow \delta^{p-2} C_*(X) \xrightarrow{\delta} \delta^{p-1} C_*(X) \rightarrow 0. \end{aligned}$$

These induce long exact sequences in homology. All homology is finitely generated and torsion over the PID  $\mathbf{F}_p[t, t^{-1}]$ . We use shorthand notation – if  $\rho \in \mathbf{F}_p[G]$ , we write  $\chi^\rho(X)$  instead of  $\chi(H_*(\rho C_*(X)))$ . The above homological considerations show

$$\begin{aligned} \chi(\bar{X}) &= \chi(X^G)\chi^\sigma(X) \\ \chi(X) &= \chi^\delta(X)\chi(X^G)\chi^\sigma(X) \\ \chi^\delta(X) &= \chi^\sigma(X)\chi^{\delta^2}(X) \\ &\quad \vdots \\ \chi^{\delta^{p-2}}(X) &= \chi^\sigma(X)\chi^\sigma(X). \end{aligned}$$

Multiplying all equations but the first together and cancelling terms we see

$$\chi(X) = \chi(X^G) \cdot \chi^\sigma(X)^p.$$

Using the first equation to substitute for  $\chi^\sigma(X)$  one finds

$$\chi(X) = \chi(\bar{X})^p / \chi(X^G)^{p-1}.$$

Finally suppose  $G$  has order  $p^r$ . Let  $G_1$  be a normal subgroup of index  $p$ . By the exact sequences above  $\text{rk } H_*(X/G_1; \mathbf{F}_p) < \infty$ . By applying inductively the result for the  $G_1$ -action on  $X$  and the  $G/G_1$  action on  $X/G_1$ , Theorem B follows.

## §2. HIGH-DIMENSIONAL PERIODIC KNOTS

One advantage of our approach to Murasugi's congruence is that it applies equally well to a more general situation. Higher-dimensional periodic knots

were introduced in the thesis of *R. Cruz* [C]. He showed that if there is a semifree  $\mathbf{Z}/q$ -action on  $S^n$  with non-empty fixed set and an invariant knot  $K^{n-2}$  disjoint from the fixed set, then the fixed set is  $S^1$  if  $q \neq 2$ , and is  $S^1$  or  $S^0$  if  $q = 2$ .

For our purposes a knot  $K$  in a homology  $n$ -sphere  $\Sigma$  is an embedded  $(n - 2)$ -dimensional homology sphere. Let  $G$  be a finite group. The knot  $K$  is *G-periodic* if it is invariant under a semifree  $G$ -action on  $\Sigma$  with fixed set  $B \cong S^1$  disjoint from  $K$ . To simplify technicalities we assume the action is smooth. Several complications arise: the group need not be cyclic, the action need not be linear and the quotient  $\bar{\Sigma} = \Sigma/G$  will not be a manifold. (Even in the linear case the quotient looks like a double suspension of a spherical space form.) However we can still make sense of Alexander polynomials.

PROPOSITION 2.1.  $H_*(\bar{\Sigma} - \bar{K}) \cong H_*(S^1)$ .

First we need a lemma.

LEMMA 2.2. *The linking number  $\lambda = \text{lk}(B, K)$  is relatively prime to the order of  $G$ .*

*Proof.* (See also [C, 2.1.1]). By restricting the action to a subgroup  $\mathbf{Z}/p$  of  $G$ , we will assume  $G = \mathbf{Z}/p$ , and show  $(\lambda, p) = 1$ . By applying the Lefschetz Fixed-Point Theorem to a generator  $g$  of  $\mathbf{Z}/p$ , we see that if  $n$  is odd, the action on  $K$  is orientation-preserving, while if  $n$  is even, then  $p = 2$  and the action is orientation-reversing. For local coefficients we will use  $\mathbf{Z}^t$ , the integers with the  $\mathbf{Z}[\mathbf{Z}/p]$ -module structure given by  $(\sum a_i g^i) \cdot k = \sum a_i (-1)^{i(n+1)} k$ .

Let  $\bar{\Sigma} - B \rightarrow K(\mathbf{Z}/p, 1)$  classify the  $G$ -cover. We will consider the commutative diagram:

$$\begin{array}{ccccc}
 H_{n-2}(K; \mathbf{Z}) & \xrightarrow{\alpha} & H_{n-2}(\bar{K}; \mathbf{Z}^t) & \rightarrow & H_{n-2}(K(\mathbf{Z}/p, 1); \mathbf{Z}^t) \\
 (*) & & \downarrow & & \parallel \\
 & & H_{n-2}(\Sigma - B; \mathbf{Z}) & \rightarrow & H_{n-2}(\bar{\Sigma} - B; \mathbf{Z}^t) \rightarrow H_{n-2}(K(\mathbf{Z}/p, 1); \mathbf{Z}^t) .
 \end{array}$$

The two groups on the left are infinite cyclic and the left vertical map is multiplication by  $\lambda$ . A diagram chase shows we will be done if we can show both horizontal exact sequences are isomorphic to the short exact sequence  $0 \rightarrow \mathbf{Z} \rightarrow \mathbf{Z} \rightarrow \mathbf{Z}/p \rightarrow 0$ .

The map  $\alpha$  is isomorphic to  $\mathbf{Z} \xrightarrow{\times p} \mathbf{Z}$  because it comes from a  $p$ -fold cover of  $(n - 2)$ -dimensional closed manifolds. The map

$$H_{n-2}(\bar{K}; \mathbf{Z}^t) \rightarrow H_{n-2}(\mathbf{Z}/p; \mathbf{Z}^t)$$

we compute algebraically by using a free  $\mathbf{Z}G$ -resolution of  $\mathbf{Z}$  as a substitute for the Eilenberg-MacLane space. By lifting a CW structure on  $\bar{K}$  to  $K$ ,

$$C_*(K) = \{C_{n-2} \rightarrow \dots \rightarrow C_0\}$$

with the  $i$ -chains  $C_i$  free  $\mathbf{Z}G$ -modules. By mapping a free  $\mathbf{Z}G$ -module onto  $\ker(C_{n-2} \rightarrow C_{n-3})$  and continuing inductively, one constructs a free  $\mathbf{Z}G$ -resolution of  $\mathbf{Z}$

$$D_* = \{\dots \rightarrow D_n \rightarrow D_{n-1} \rightarrow C_{n-2} \rightarrow \dots \rightarrow C_0\}.$$

It follows that

$$H_{n-2}(\bar{K}; \mathbf{Z}^t) = H_{n-2}(C_*(K) \otimes_{\mathbf{Z}G} \mathbf{Z}^t)$$

maps onto  $H_{n-2}(D_* \otimes_{\mathbf{Z}G} \mathbf{Z}^t) = H_{n-2}(\mathbf{Z}/p; \mathbf{Z}^t)$ . Furthermore by using the standard  $\mathbf{Z}G$ -resolution of  $\mathbf{Z}$  (see e.g. [Mac]), one easily computes that  $H_{n-2}(\mathbf{Z}/p; \mathbf{Z}^t) \cong \mathbf{Z}/p$ .

Choose a  $G$ -invariant normal disk to  $B$  in  $\Sigma$  and let  $S^{n-2}$  be its boundary. Then the inclusion  $S^{n-2} \rightarrow \Sigma - B$  is a homology equivalence. By the comparison theorem applied to the spectral sequence of the  $G$ -coverings (see [Mac]), the bottom row of (\*) is isomorphic to

$$H_{n-2}(S^{n-2}; \mathbf{Z}) \rightarrow H_{n-2}(S^{n-2}/G; \mathbf{Z}^t) \rightarrow H_{n-2}(G; \mathbf{Z}^t),$$

and hence by the previous paragraph to  $0 \rightarrow \mathbf{Z} \rightarrow \mathbf{Z} \rightarrow \mathbf{Z}/p \rightarrow 0$ . Thus  $(\lambda, p) = 1$ .

*Proof of 2.1.* Let  $N$  be an equivariant tubular neighborhood of  $B$ . Then

$$0 = H_*(\Sigma - K, N; \mathbf{Z}[1/\lambda]) = H_*(\Sigma - K - B, N - B; \mathbf{Z}[1/\lambda])$$

where the first equality holds by the definition of linking number and the second by excision. Then

$$\begin{aligned} 0 &= H_*((\Sigma - K - B)/G, (N - B)/G; \mathbf{Z}[1/\lambda]) = H_*((\Sigma - K)/G, N/G; \mathbf{Z}[1/\lambda]) \\ &= H_*((\Sigma - K)/G, B; \mathbf{Z}[1/\lambda]), \end{aligned}$$

where the first equality follows from the spectral sequence of a covering, the second by excision and the third by the homotopy equivalence  $B \rightarrow N/G$ . Thus  $H_*(\bar{\Sigma} - \bar{K})$  looks like  $H_*(S^1)$  except possibly for some  $\lambda$ -torsion. But by 2.1,  $\lambda$  is prime to the order of  $G$ , so for all primes  $q$  dividing  $\lambda$ , the transfer map  $\text{tr}: H_*(\bar{\Sigma} - \bar{K}; \mathbf{Z}/q) \rightarrow H_*(\Sigma - K; \mathbf{Z}/q)$  is injective so there is no extra  $\lambda$ -torsion.

To state Murasugi's congruence in higher dimensions is it necessary to find a substitute for the Alexander polynomial. Let  $X$  and  $\bar{X}$  be the infinite cyclic

covers of  $\Sigma - K$  and  $\bar{\Sigma} - \bar{K}$  respectively. Let  $\Delta_K(t) = \prod_{i>0} [H_i(X)]^{(-1)^{i+1}}$  and  $\Delta_{\bar{K}}(t) = \prod_{i>0} [H_i(\bar{X})]^{(-1)^{i+1}}$ . The Wang sequence shows that multiplication by  $t - 1$  induces an isomorphism on  $H_i(X)$  for  $i > 0$ , so that if we take the polynomial represented by  $[H_i(X)]$  and plug in  $t = 1$  we get  $\pm 1$ . (Indeed if we consider the ring homomorphism  $\varphi: \mathbf{Z}[t, t^{-1}] \rightarrow \mathbf{Z}$  defined by  $\varphi(t) = 1$ , then  $\varphi([H_i(X)])$  is a divisor of  $[H_i(X) \otimes_{\mathbf{Z}[t, t^{-1}]} \mathbf{Z}] = [0] = 1 \in \mathbf{Z}/\mathbf{Z}^*$ .) Thus  $[H_i(X)]$  represented a non-zero element in  $\mathbf{F}_p[t, t^{-1}]$ , and hence  $\Delta_K(t)$  and  $\Delta_{\bar{K}}(t)$  give well-defined elements of  $\mathbf{F}_p(t)^*/\mathbf{F}_p[t, t^{-1}]^*$ . Then the considerations of §1 show:

**THEOREM 2.3.** *Let  $K$  be a  $G$ -periodic knot in a homology  $q$ -sphere  $\Sigma$  with fixed set  $B$ , where  $G$  is a group of prime power order  $p^r$ . Let  $\lambda$  be the linking number of  $K$  and  $B$ . Then*

$$\Delta_K(t) \equiv \Delta_{\bar{K}}(t)^{p^r} (1 + t + \dots + t^{\lambda-1})^{p^r-1} \pmod{p} .$$

### §3. AN APPLICATION OF MURASUGI'S CONGRUENCE

For any  $\lambda \equiv \pm 1 \pmod{8}$ , T. tom Dieck and J. Davis [D-D] constructed a 2-component link with linking number  $\lambda$  in a homology 3-sphere  $\Omega$  whose  $C_2 \times C_2$ -cover branched over the link is a homology 3-sphere  $\Sigma$ . We will show that this congruence condition is necessary. Equivalently, we show

**THEOREM 3.1.** *Suppose the Klein 4-group  $G \times H \cong C_2 \times C_2$  acts on a homology 3-sphere  $\Sigma$  so that the fixed sets  $\Sigma^G$  and  $\Sigma^H$  are disjoint circles. Then their linking number  $\lambda$  is congruent to  $\pm 1$  modulo 8.*

*Proof.* We have

$$\begin{array}{ccc} \Sigma & \rightarrow & \Sigma/G \\ \downarrow & & \downarrow \\ \Sigma/H & \rightarrow & \Sigma/(G \times H) . \end{array}$$

All four of these manifolds are homology 3-spheres and each has two disjoint circles given by the images of the fixed sets. The linking numbers of each pair of circles are all equal.

Let  $K = \Sigma^G/G \subset \Sigma/G$  and  $\bar{K} = K/H \subset \Sigma/(G \times H)$ . Then  $K$  is a knot of period 2. Renormalize  $\Delta_K(t)$  and  $\Delta_{\bar{K}}(t) \in \mathbf{Z}[t, t^{-1}]$  so that  $\Delta_K(t) = \Delta_K(t^{-1})$ ,  $\Delta_{\bar{K}}(t) = \Delta_{\bar{K}}(t^{-1})$ , and  $\Delta_K(1) = 1 = \Delta_{\bar{K}}(1)$ . Murasugi's congruence shows