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covers of $\Sigma - K$ and $\bar{\Sigma} - \bar{K}$ respectively. Let $\Delta_K(t) = \Pi_{i>0}[H_i(X)]^{(-1)^{i+1}}$ and $\Delta_{\bar{K}}(t) = \Pi_{i>0}[H_i(\bar{X})]^{(-1)^{i+1}}$. The Wang sequence shows that multiplication by $t-1$ induces an isomorphism on $H_i(X)$ for $i > 0$, so that if we take the polynomial represented by $[H_i(X)]$ and plug in $t = 1$ we get $\pm 1$. (Indeed if we consider the ring homomorphism $\varphi: \mathbf{Z}[t, t^{-1}] \to \mathbf{Z}$ defined by $\varphi(t) = 1$, then $\varphi([H_i(X)])$ is a divisor of $[H_i(X) \otimes_{\mathbf{Z}[t,t^{-1}]} \mathbf{Z}] = [0] = 1 \in \mathbf{Z}/\mathbf{Z}^*$.) Thus $[H_i(X)]$ represented a non-zero element in $\mathbf{F}_p[t, t^{-1}]$, and hence $\Delta_K(t)$ and $\Delta_{\bar{K}}(t)$ give well-defined elements of $\mathbf{F}_p(t)^*/\mathbf{F}_p[t, t^{-1}]^*$. Then the considerations of §1 show:

THEOREM 2.3. *Let $K$ be a G-periodic knot in a homology q-sphere $\Sigma$ with fixed set $B$, where $G$ is a group of prime power order $p^r$. Let $\lambda$ be the linking number of $K$ and $B$. Then*

$$\Delta_K(t) \doteq \Delta_{\bar{K}}(t)^{p^r}(1 + t + \ldots + t^{\lambda-1})^{p^r-1} \pmod{p} .$$

## §3. AN APPLICATION OF MURASUGI'S CONGRUENCE

For any $\lambda \equiv \pm 1 \pmod 8$, T. tom Dieck and J. Davis [D-D] constructed a 2-component link with linking number $\lambda$ in a homology 3-sphere $\Omega$ whose $C_2 \times C_2$-cover branched over the link is a homology 3-sphere $\Sigma$. We will show that this congruence condition is necessary. Equivalently, we show

THEOREM 3.1. *Suppose the Klein 4-group $G \times H \cong C_2 \times C_2$ acts on a homology 3-sphere $\Sigma$ so that the fixed sets $\Sigma^G$ and $\Sigma^H$ are disjoint circles. Then their linking number $\lambda$ is congruent to $\pm 1$ modulo 8.*

*Proof.* We have

$$\begin{array}{ccc} \Sigma & \to & \Sigma/G \\ \downarrow & & \downarrow \\ \Sigma/H & \to & \Sigma/(G \times H) . \end{array}$$

All four of these manifolds are homology 3-spheres and each has two disjoint circles given by the images of the fixed sets. The linking numbers of each pair of circles are all equal.

Let $K = \Sigma^G/G \subset \Sigma/G$ and $\bar{K} = K/H \subset \Sigma/(G \times H)$. Then $K$ is a knot of period 2. Renormalize $\Delta_K(t)$ and $\Delta_{\bar{K}}(t) \in \mathbf{Z}[t, t^{-1}]$ so that $\Delta_K(t) = \Delta_K(t^{-1})$, $\Delta_{\bar{K}}(t) = \Delta_{\bar{K}}(t^{-1})$, and $\Delta_K(1) = 1 = \Delta_{\bar{K}}(1)$. Murasugi's congruence shows

$$(**) \qquad \Delta_K(t) = \Delta_{\bar{K}}(t)^2(t^{(1-\lambda)/2} + \ldots + 1 + \ldots + t^{(\lambda-1)/2}) + 2f(t)\,,$$

where $f(t) \in \mathbf{Z}[t, t^{-1}]$ satisfies $f(t) = f(t^{-1})$. Writing

$$f(t) = a_n t^{-n} + \ldots + a_0 + \ldots + a_n t^n\,,$$

we see $f(1) \equiv f(-1) \pmod 4$. Since $\Sigma \to \Sigma/G$ is a 2-fold cover branched over $K$, $|\Delta_K(-1)| = |H_1(\Sigma)| = 1$. So $1 = \Delta_K(1) \equiv \Delta_K(-1) \pmod 4$, and we see $\Delta_K(-1) = 1$. Take equation (**) and plug in $t = 1$ and $t = -1$:

$$1 = 1 \cdot \lambda + 2 \cdot f(1)$$
$$1 = 1 \cdot (-1)^{(\lambda-1)/2} + 2 \cdot f(-1)\,.$$

Thus $\lambda \equiv (-1)^{(\lambda-1)/2} \pmod 8$ so $\lambda \equiv \pm 1 \pmod 8$.

Applying the high-dimensional version of Murasugi's congruence ones sees that if $G \times H \cong C_2 \times C_2$ acts on a homology $q$-sphere $\Sigma$ so that $\Sigma^G$ is a homology $q-2$ sphere and $\Sigma^H$ is a circle disjoint from $\Sigma^G$, then their linking number $\lambda$ is congruent to $\pm 1$ modulo 8. This and considerations from $L$-theory lead us to conjecture that if $G \times H \cong C_2 \times C_2$ acts on a homology $q$-sphere $\Sigma$ so that $\Sigma^G$ is a homology $k$-sphere and $\Sigma^H$ is a homology $q-k-1$-sphere disjoint from $\Sigma^G$, then their linking number $\lambda$ is congruent to $\pm 1$ modulo 8.

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