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Autor: Oprea, John
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with classifying element (over the rationals) $xy \in H^2(T^2; \mathbf{Q})$, where x and y are one-dimensional generators. The minimal model of M is then given by

$$\Lambda(M) = \Lambda(x, y, z) \quad \deg(x) = \deg(y) = \deg(z) = 1$$

with $dx = 0 = dy$ and $dz = xy$. Additive generators for cohomology are then,

$H^1: x, y$

$H^2: xz, yz$ (Massey products!)

$H^3: zyx$.

Note that $\mathrm{cup}(M) = 2$, but $\mathrm{cat}(M) = 3$.

In some sense then, the proof of Theorem 1 is simply an observation that the techniques of rational homotopy theory work particularly well for nilmanifolds.

PROBLEM. If π is not nilpotent, then a $K(\pi, 1)$ is not a nilpotent space, so the minimal model does not describe a "rational type". Is it possible, however, that enough information about a $K(\pi, 1)$ is present in the model to determine its category (in the compact case say)?

§5. HIGHER DEGREE ANALOGUES

An analogue of the minimal model of a nilmanifold is one of the form,

$$(\Lambda(x_1, \cdots x_n), d) , \quad \mathrm{degree}(x_i) = \mathrm{odd} .$$

Such an algebra is known to satisfy rational Poincaré duality (see [5]) and to have formal top dimension $\sum_i \deg(x_i)$. But, plainly, the same argument as before applies to show that the "only" element in this exterior algebra which can reach the stated dimension is $x_1 \cdots x_n$. Hence (since this is the longest product in Λ), the fundamental class is maximally represented by a product of length n and

LEMMA. $e_0(\Lambda) = n$.

Now, we may consider Λ as built up by adjoining odd generators one at a time (with decomposable differential). Let ΛZ be a minimal cdga and y of odd degree. Then

PROPOSITION. (See Theorem 4.7 and Lemma 6.6 of [3].)

$$\mathrm{cat}_0(\Lambda Z \otimes \Lambda y) \leqslant \mathrm{cat}_0(\Lambda Z) + 1 .$$

Proof. Suppose $\mathrm{cat}_0(\Lambda Z) = m$. Then ΛZ is a retract of $\Lambda Z/\Lambda^{>m}Z$ and we see that $\Lambda Z \otimes \Lambda y$ is a retract of $\Lambda Z/\Lambda^{>m}Z \otimes \Lambda y$. Now, the maximal product length of $\Lambda Z/\Lambda^{>m}Z \otimes \Lambda y$ is $m+1$ and this is sufficient to ensure $\mathrm{cat}_0(\Lambda Z \otimes \Lambda y) \leqslant m+1$. □

Now, by induction, we see that $\mathrm{cat}_0(\Lambda) \leqslant n$ (since for x_1 of odd degree $\mathrm{cat}_0(\Lambda x_1) = 1$). Putting this together with the Lemma gives

THEOREM 2. *If* $\Lambda = (\Lambda(x_1, \cdots x_n), d)$ *with* $\deg(x_i) = \mathrm{odd}$ *for each* i, *then* $\mathrm{cat}_0(\Lambda) = n$.

This result may be applied, for example, to a manifold obtained as an iterated principal bundle. That is, for compact Lie groups G_i, $i = 1$ to N.

$M_1 = G_1$; M_i is obtained from M_{i-1} as a principal G_i-bundle over M_{i-1}.

$M = M_N$

Each G_i is, rationally, a product of $\mathrm{rank}(G_i)$ odd spheres, so the minimal model of M has the form,

$$\Lambda(M) = (\Lambda(x_1, \cdots x_s), d)$$

with $\deg(x_i) = \mathrm{odd}$ and $s = \sum_{i=1}^{N} \mathrm{rank}(G_i)$.

COROLLARY. $\mathrm{cat}_0(M) = \sum_{i=1}^{N} \mathrm{rank}(G_i)$.

COROLLARY. *If* M *is an iterated principal bundle with fibres* G_i, *then the number of critical points of any smooth function on* M *is bounded below by* $\sum_i \mathrm{rank}(G_i) + 1$.

Note that we have not determined $\mathrm{cat}(M)$, so the true effectiveness of Lusternik-Schnirelmann theory may not have been exploited.

§6. GANEA'S CONJECTURE

The Ganea Conjecture states that, for a finite CW complex X, $\mathrm{cat}(X \times S^k) = \mathrm{cat}(X) + 1$ for any sphere S^k. Although unproven in general, various cases of the conjecture have been shown to be true. We add nilmanifolds to that list:

THEOREM. *Ganea's Conjecture is true for nilmanifolds.*