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Thus the equality (1.1) holds in that case.

2) Assume

$$W = \langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^3 = (s_1 s_3)^3 = (s_2 s_3)^3 = 1 \rangle$$

to be a Coxeter group of type  $\tilde{A}_2$ . The geometric realisation of  $\Sigma(W, S)$  is a plane. We have

$$\begin{aligned} W_S(t_1, t_2) &= \frac{(1+t_1)(1+t_1+t_1^2)}{(1-t_1)(1-t_1^2)} + 3 \frac{1+t_1+t_1^2}{(1-t_1)(1-t_1^2)} t_2 \\ &\quad + 3 \frac{1}{(1-t_1)(1-t_1^2)} t_2^2 + t_2^3, \end{aligned}$$

and

$$\begin{aligned} W_S(t) &= \frac{(1+t)(1+t+t^2)}{(1-t)(1-t^2)}, \\ W_{\{s_1, s_2\}}(t) &= W_{\{s_1, s_3\}}(t) = W_{\{s_2, s_3\}}(t) = (1+t)(1+t+t^2), \\ W_{\{s_1\}}(t) &= W_{\{s_2\}}(t) = W_{\{s_3\}}(t) = 1+t, \\ W_\emptyset(t) &= 1. \end{aligned}$$

Thus the equality (1.1) holds in that case.

In Section 2 we will recall some definitions in the theory of Coxeter complexes, we will define the complex growth series of a Coxeter system  $(W, S)$ , we will prove that  $W_S(t_1, 0) = W_S(t_1)$  and  $W_S(0, t_2) = (1+t_2)^{|S|}$  (Proposition 1), we will prove the equalities (1.4) and (1.5) (Proposition 2), and we will prove the Main Theorem.

## 2. COMPLEX GROWTH SERIES

We assume the reader to be familiar with the notions of simplicial complex, chamber complex, adjacency between two chambers, gallery and labelling. We refer to [2, Chap. I, Appendix] for a good exposition of these notions.

Let  $(W, S)$  be a Coxeter system. A *special coset* of  $(W, S)$  is a coset  $wW_X$ , with  $w \in W$  and  $X \subseteq S$ . We denote by  $\Sigma = \Sigma(W, S)$  the poset of all special cosets, ordered by the reverse inclusion;  $B \leq A$  in  $\Sigma$  if  $B \supseteq A$  in  $W$ . The poset  $\Sigma$  is a labelled chamber simplicial complex (see [2, Chap. III, §1]).

A *chamber* of  $\Sigma$  is a singleton  $\{w\}$  with  $w \in W$ . A *vertex* of  $\Sigma$  is a special coset  $wW_{S-\{s\}}$  with  $w \in W$  and  $s \in S$ . The face of  $\Sigma$  of dimension  $-1$  is the

coset  $1 \cdot W = W$  (this face has 0 vertices). The *fundamental chamber* of  $\Sigma$  is  $\{1\}$ .

The Coxeter group  $W$  naturally acts on  $\Sigma$  by

$$(2.1) \quad w(vW_X) = (wv)W_X,$$

where  $w \in W$ , and  $vW_X$  is a face of  $\Sigma$  (i.e. a special coset).

The map which associates to a face  $F = wW_X$  the subset  $\lambda(F) = S - X$  of  $S$  determines a labelling on  $\Sigma$ , called the *canonical labeling* of  $\Sigma$ , where  $\lambda(F)$  is the *type* of a face  $F$ .

Two chambers  $\{w\} \neq \{w'\}$  are *adjacent* if they have a common codimension 1 face, namely, if there exists an  $s \in S$  such that  $w' = ws$ . A *gallery of length  $d$*  is a sequence  $\{C_i\}_{i=0}^d$  of  $d + 1$  chambers such that  $C_i$  and  $C_{i+1}$  are adjacent for  $i = 0, 1, \dots, d - 1$ . In fact, to give a gallery  $\{C_i\}_{i=0}^d$  is equivalent to give a source chamber  $C_0$  and a sequence  $s_1, \dots, s_d$  of elements of  $S$ ; the equivalence is given by  $C_i = s_i \dots s_2 s_1(C_0)$ . A gallery  $\{C_i\}_{i=1}^d$  joining two chambers  $C_0$  and  $C_d$  is called *minimal* if there is no gallery joining  $C_0$  and  $C_d$  with a smaller length.

The *distance*  $d(C, D)$  between two chambers  $C$  and  $D$  is the length of a minimal gallery joining  $C$  and  $D$ . We can easily see that, if  $C = \{w\}$  and  $D = \{v\}$ , then

$$(2.2) \quad d(C, D) = l(w^{-1}v).$$

The *distance*  $d(C, F)$  between a chamber  $C$  and a face  $F$  of  $\Sigma$  is

$$(2.3) \quad d(C, F) = \min \{d(C, D) \mid D \text{ a chamber having } F \text{ as face}\}.$$

As in (2.2), if  $C = \{w\}$  and  $F = vW_X$ , then

$$(2.4) \quad d(C, F) = \min \{l(u) \mid u \in w^{-1}vW_X\}.$$

The *complex growth series* of a Coxeter system  $(W, S)$  is the formal series in two variables

$$(2.5) \quad W_S(t_1, t_2) = \sum_F t_1^{d(C_0, F)} t_2^{\text{codim}(F)},$$

where the sum is over all the faces  $F$  of  $\Sigma$ , and where  $C_0 = \{1\}$  is the fundamental chamber.

Before stating and proving Propositions 1 and 2 and the Main Theorem, we are going to state two known results (Lemmas 1 and 2). A proof of Lemma 1 can be found either in [1, §4.1, exercise 3] or in [3, Lemma 1]. A proof of Lemma 2 can be found in [2, Chap. IV, §6].

Let  $X \subseteq S$  be a subset and let  $v \in W$ . The element  $v$  is called  $X$ -minimal if  $v$  is of minimal length among the elements of  $vW_X$ .

LEMMA 1. Let  $X \subseteq S$  be a subset and let  $v \in W$  be an  $X$ -minimal element of  $W$ . Then

- i)  $v$  is the unique  $X$ -minimal element of  $vW_X$ ,
- ii) for every  $w = vu \in vW_X$ , with  $u = v^{-1}w \in W_X$ , one has  $l(w) = l(v) + l(u)$ .

For an integer  $d \geq 0$ , we denote by  $\Sigma_d$  the subcomplex of  $\Sigma = \Sigma(W, S)$  generated by the chambers  $C$  of  $\Sigma$  at distance  $\leq d$  of  $C_0 = \{1\}$ .

$$\Sigma_d = \bigcup_F F,$$

where the union is over all the faces  $F$  of  $\Sigma$  such that  $d(C_0, F) \leq d$ . We denote by  $|\Sigma_d|$  the geometric realization of  $\Sigma_d$ .

LEMMA 2. i) Let  $(W, S)$  be a finite Coxeter system. Set  $m = \max_{w \in W} l(w)$ . Then  $|\Sigma_d|$  is contractible if  $d < m$ , and  $|\Sigma_d|$  is homotopic to the sphere  $S^{|S|-1}$  of dimension  $|S| - 1$  if  $d \geq m$ .

ii) Let  $(W, S)$  be an infinite Coxeter system. Then  $|\Sigma_d|$  is contractible.

PROPOSITION 1. Let  $(W, S)$  be a Coxeter system. Then

$$(2.6) \quad W_S(t_1, 0) = W_S(t_1) \quad \text{and}$$

$$(2.7) \quad W_S(0, t_2) = (1 + t_2)^{|S|}.$$

*Proof.*

$$W_S(t_1, 0) = \sum_F t_1^{d(C_0, F)},$$

where the sum is over all the faces of  $\Sigma$  of codimension 0, i.e. over all the chambers of  $\Sigma$ . Furthermore, if  $F = C = \{w\}$ , then, by (2.2),  $d(C_0, F) = l(w)$ . It follows that

$$W_S(t_1, 0) = \sum_{w \in W} t_1^{l(w)} = W_S(t_1).$$

Now,

$$W_S(0, t_2) = \sum_F t_2^{\text{codim}(F)},$$

where the sum is over all the faces  $F$  of  $\Sigma$  at distance 0 of  $C_0$ , i.e. over all the faces of  $C_0$ . Since  $C_0$  is an  $|S| - 1$  dimensional simplex, it has  $\binom{|S|}{i}$  faces of dimension  $i$  (where  $i = 0, 1, \dots, |S|$ ). It follows that

$$W_S(0, t_2) = \sum_{i=0}^{|S|} \binom{|S|}{i} t_2^i = (1 + t_2)^{|S|}. \quad \square$$

PROPOSITION 2. i) Let  $(W, S)$  be a finite Coxeter system. Then

$$(2.8) \quad W_S(t_1, -1) = t_1^m,$$

where  $m$  is the maximal length in  $W$ .

ii) Let  $(W, S)$  be an infinite Coxeter system. Then

$$(2.9) \quad W_S(t_1, -1) = 0.$$

*Proof.* Recall that  $\Sigma_d$  is the subcomplex of  $\Sigma$  generated by the chambers of  $\Sigma$  at distance  $\leq d$  of  $C_0$ , and that  $|\Sigma_d|$  is the geometric realization of  $\Sigma_d$ . We denote by  $E(|\Sigma_d|)$  the Euler characteristic of  $|\Sigma_d|$ . It is well known that  $E(|\Sigma_d|)$  can be computed as follows:

$$\begin{aligned} (-1)^{|S|-1} E(|\Sigma_d|) &= (-1)^{|S|-1} \sum_{\substack{d(C_0, F) \leq d \\ F \neq W}} (-1)^{\dim(F)} \\ &= \sum_{\substack{d(C_0, F) \leq d \\ F \neq W}} (-1)^{\text{codim}(F)}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} W_S(t_1, -1) &= \sum_F t_1^{d(C_0, F)} (-1)^{\text{codim}(F)} \\ &= \sum_{d=0}^{\infty} \left( \sum_{d(C_0, F)=d} (-1)^{\text{codim}(F)} \right) t_1^d. \end{aligned}$$

Thus

$$(2.10) \quad (-1)^{|S|-1} (E(|\Sigma_d|) - E(|\Sigma_{d-1}|))$$

is the coefficient of  $t_1^d$  in  $W_S(t_1, -1)$  for  $d \geq 1$ , and

$$(2.11) \quad (-1)^{|S|-1} E(|\Sigma_0|) + (-1)^{|S|}$$

is the coefficient of  $t_1^0$  in  $W_S(t_1, -1)$ . Lemma 2 implies that, if  $(W, S)$  is a finite Coxeter system, then

$$E(|\Sigma_d|) = \begin{cases} 1 & \text{if } d < m, \\ 1 + (-1)^{|S|-1} & \text{if } d \geq m, \end{cases}$$

where  $m$  is the maximal length in  $W$ ; and if  $(W, S)$  is an infinite Coxeter system, then

$$E(|\Sigma_d|) = 1,$$

for all  $d \geq 0$ . Replacing  $E(|\Sigma_d|)$  by its value in (2.10) and (2.11), we obtain the equalities (2.8) and (2.9).  $\square$

MAIN THEOREM. *Let  $(W, S)$  be a Coxeter system. Then*

$$(2.12) \quad W_S(t_1, t_2) = \sum_{X \subseteq S} t_2^{|X|} \frac{W_S(t_1)}{W_X(t_1)}.$$

*Proof.* Recall that the map which associates to a face  $F = wW_X$  the subset  $\lambda(F) = S - X$  of  $S$  determines a labelling on  $\Sigma$ , where  $\lambda(F)$  is the type of the face  $F$ . Clearly, if  $\lambda(F) = Y$ , then  $\dim(F) = |Y| - 1$  and  $\text{codim}(F) = |S| - |Y| = |S - Y|$ . Therefore

$$(2.13) \quad W_S(t_1, t_2) = \sum_{Y \subseteq S} t_2^{|S-Y|} \left( \sum_{F \in \mathcal{F}_Y} t_1^{d(C_0, F)} \right),$$

where  $\mathcal{F}_Y$  is the set of faces of  $\Sigma$  of type  $Y$ . Let us prove

$$(2.14) \quad \sum_{F \in \mathcal{F}_Y} t_1^{d(C_0, F)} = \frac{W_S(t_1)}{W_{S-Y}(t_1)},$$

for every  $Y \subseteq S$ . The equalities (2.13) and (2.14) clearly imply (2.12).

Let  $X = S - Y$ . Recall that an element  $v \in W$  is  $X$ -minimal if it is of minimal length in  $vW_X$ . Every face  $F \in \mathcal{F}_Y$  can be written  $F = vW_X$  with  $v$   $X$ -minimal (take any element of minimal length in  $F$ ). By (2.4), we have

$$d(C_0, F) = l(v).$$

Lemma 1 shows that, for every  $F \in \mathcal{F}_Y$ , there is an unique  $X$ -minimal element  $v$  in  $F$ . Therefore

$$(2.15) \quad \sum_{F \in \mathcal{F}_Y} t_1^{d(C_0, F)} = \sum_{v \in A_X} t_1^{l(v)},$$

where  $A_X$  is the set of all the  $X$ -minimal elements of  $W$ . Finally, Lemma 1 shows

$$\begin{aligned} W_S(t_1) &= \sum_{w \in W} t_1^{l(w)} \\ &= \sum_{v \in A_X} \sum_{w \in vW_X} t_1^{l(w)} \quad (\text{Lemma 1.i}) \\ &= \sum_{v \in A_X} \sum_{u \in W_X} t_1^{l(v)+l(u)} \quad (\text{Lemma 1.ii}) \\ &= \left( \sum_{v \in A_X} t_1^{l(v)} \right) W_X(t_1). \end{aligned}$$

This and (2.15) imply (2.14)  $\square$

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