

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 38 (1992)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SIMPLE PROOF OF A THEOREM OF THUE ON THE MAXIMAL DENSITY OF CIRCLE PACKINGS IN E^2
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Kapitel: Remark on the uniqueness of the finite packings of maximal density
DOI: <https://doi.org/10.5169/seals-59487>

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REMARK ON THE UNIQUENESS OF THE FINITE PACKINGS
OF MAXIMAL DENSITY

All natural or practical examples of circle packings such as bees living in a honeycomb or a bundle of fibre-glass optical tubes are always packing problems of a finite number of circles (i.e. packings of their cross-section circles). The infinite circle packings of the entire plane of E^2 are actually the limit situation of the finite circle packings. Therefore, it is natural to give an appropriate definition of the concept of global density for a finite circle packing. We propose the following definition of a cluster of circles and the (global) density of a cluster of circles, namely

Definition. A packing of finite number of equal circles is called a *cluster of circles* if any two of them can be linked through neighboring pairs of center distances less than $2\sqrt{2}$ times the radii.

Let \mathcal{C} be a given cluster of circles. Then, an extension, \mathcal{C}^* , of \mathcal{C} is called a *saturated coating of \mathcal{C}* if all circles of $\mathcal{C}^* \setminus \mathcal{C}$ are neighbors of some circles in \mathcal{C} and it is impossible to add any more such neighbors to \mathcal{C}^* . Observe that *every circle in \mathcal{C} has a saturated set of neighbors in \mathcal{C}^** and hence has a well-defined *local cell* with respect to \mathcal{C}^* . The *usual weighted average* of all the local densities of circles in \mathcal{C} with respect to the given saturated coating \mathcal{C}^* is defined to be the density of \mathcal{C} in \mathcal{C}^* , i.e. $\rho(\mathcal{C} \text{ rel } \mathcal{C}^*)$.

Definition: The global density of \mathcal{C} is defined to be the least upper bound of the densities of \mathcal{C} in all possible saturated coatings of \mathcal{C} , namely

$$\rho(\mathcal{C}) = \text{l.u.b.} \{ \rho(\mathcal{C} \text{ rel } \mathcal{C}^*) \}$$

where \mathcal{C}^* run through all possible saturated coatings of \mathcal{C} .

UNIQUENESS THEOREM (On finite circle packings of maximal density). $\pi/\sqrt{12}$ is still the maximal possible global density of all clusters of circles, and the global density of a cluster of circles, \mathcal{C} , attains the above maximum of $\pi/\sqrt{12}$ when and only when \mathcal{C} is a subcluster of circles in the hexagon packing.

Proof. It is again a direct consequence of the above Theorem on the maximal local density and its uniqueness.