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Artikel: SIMPLE PROOF OF A THEOREM OF THUE ON THE MAXIMAL DENSITY OF CIRCLE PACKINGS IN E^2

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REMARK ON THE UNIQUENESS OF THE FINITE PACKINGS OF MAXIMAL DENSITY

All natural or practical examples of circle packings such as bees living in a honeycomb or a bundle of fibre-glass optical tubes are always packing problems of a finite number of circles (i.e. packings of their cross-section circles). The infinite circle packings of the entire plane of E^2 are actually the limit situation of the finite circle packings. Therefore, it is natural to give an appropriate definition of the concept of global density for a finite circle packing. We propose the following definition of a cluster of circles and the (global) density of a cluster of circles, namely

Definition. A packing of finite number of equal circles is called a *cluster of circles* if any two of them can be linked through neighboring pairs of center distances less than $2\sqrt{2}$ times the radii.

Let $\mathscr{C}$ be a given cluster of circles. Then, an extension, $\mathscr{C}^*$, of $\mathscr{C}$ is called a *saturated coating of* $\mathscr{C}$ if all circles of $\mathscr{C}^* \setminus \mathscr{C}$ are neighbors of some circles in $\mathscr{C}$ and it is impossible to add any more such neighbors to $\mathscr{C}^*$. Observe that *every circle* in $\mathscr{C}$ has a *saturated set of neighbors in* $\mathscr{C}^*$ and hence has a well-defined *local cell* with respect to $\mathscr{C}^*$. The *usual weighted average* of all the local densities of circles in $\mathscr{C}$ with respect to the given saturated coating $\mathscr{C}^*$ is defined to be the density of $\mathscr{C}$ in $\mathscr{C}^*$, i.e. $\rho(\mathscr{C} \text{ rel } \mathscr{C}^*)$.

Definition: The global density of $\mathscr{C}$ is defined to be the least upper bound of the densities of $\mathscr{C}$ in all possible saturated coatings of $\mathscr{C}$, namely

$$\rho(\mathscr{C}) = \text{l.u.b.}\{\rho(\mathscr{C} \text{ rel } \mathscr{C}^*)\}$$

where $\mathscr{C}^*$ run through all possible saturated coatings of $\mathscr{C}$.

UNIQUENESS THEOREM (On finite circle packings of maximal density). *$\pi/\sqrt{12}$ is still the maximal possible global density of all clusters of circles, and the global density of a cluster of circles, $\mathscr{C}$, attains the above maximum of $\pi/\sqrt{12}$ when and only when $\mathscr{C}$ is a subcluster of circles in the hexagon packing.*

Proof. It is again a direct consequence of the above Theorem on the maximal local density and its uniqueness.