Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 38 (1992)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: REAL NUMBERS WITH BOUNDED PARTIAL QUOTIENTS: A SURVEY

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Kapitel: 1. Introduction and Definitions

DOI: https://doi.org/10.5169/seals-59489

L'Enseignement Mathématique, t. 38 (1992), p. 151-187

REAL NUMBERS WITH BOUNDED PARTIAL QUOTIENTS: A SURVEY

by Jeffrey SHALLIT

ABSTRACT. Real numbers with bounded partial quotients in their continued fraction expansion appear in many different fields of mathematics and computer science: Diophantine approximation, fractal geometry, transcendental number theory, ergodic theory, numerical analysis, pseudo-random number generation, dynamical systems, and formal language theory. In this paper we survey some of these applications.

1. INTRODUCTION AND DEFINITIONS

If x is a real number, we can expand x as a *simple continued fraction*

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

which we abbreviate in this paper as

$$x = [a_0, a_1, a_2, a_3, \ldots] .$$

In this paper, we only discuss the case of *regular continued fractions,* where a_0 is an integer and a_i, is a positive integer for $i \geqslant 1$; the expansion may or may not terminate. (For an introduction to continued fractions, see Hardy and Wright [135, Chap. 10]; for a more definitive work, see Perron [236]. For a history of continued fractions, see Brezinski [44].)

Supported in part by NSF Grant CCS-8817400, a Walter Burke Award from Dartmouth College and NSERC Canada.

If x is rational, then its continued fraction expansion terminates, and we can write $x = [a_0, a_1, \ldots, a_n]$. If we agree that $a_n = 1$ and $n \geqslant 1$, then this expansion is unique and we define

$$K(x) = \max_{1 \leqslant k \leqslant n} a_k ,$$

the largest partial quotient in the continued fraction for x.

If x is irrational, then its continued fraction expansion does not terminate. This expansion is unique. We write $x = [a_0, a_1, a_2, \ldots]$ and define

$$K(x) = \sup_{k \geqslant 1} a_k .$$

If $K(x) < \infty$, then we say that x has *bounded partial quotients*.

We define $\mathscr{B}_k = \{x \in \mathbf{R} \mid K(x) \leqslant k\}$, and $\mathscr{B} = \{x \in \mathbf{R} \mid K(x) < \infty\}$. Furthermore, let $\mathscr{E}_k = \mathscr{B}_k \cap (0, 1)$ and $\mathscr{E} = \mathscr{B} \cap (0, 1)$.

Real numbers with bounded partial quotients appear in many fields of mathematics and computer science: Diophantine approximation, fractal geometry, transcendental number theory, ergodic theory, numerical analysis, pseudo-random number generation, dynamical systems, and formal language theory. In this paper we survey some of these applications. Because of limited space, we cannot include a discussion of every result in detail. However, we have tried to include as complete a list of references as possible for those topics directly related to the main subject. Readers who know of other references are urged to contact the author (and provide a copy of the relevant paper, if possible). It is hoped that the list of references may contain some surprises even for experts in the field.

The author's interest in the subject arose from the material in Section 9. Because of this, the viewpoint presented in this article may be somewhat idiosyncratic.

2. Numbers of Constant Type

Let θ be an irrational number, and let $\| \theta \|$ denote the distance between θ and the closest integer.

Let $r \geqslant 1$ be a real number. We say that θ is *of type* $< r$ if

$$q \| q\theta \| \geqslant \frac{1}{r}$$

for all integers $q \geqslant 0$. Then we have the following