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$$[0, 1, 2, 2, 1, 1, 1, 1, \overbrace{2, \ldots, 2}^{8}, \overbrace{1, \ldots, 1}^{16}, \overbrace{2, \ldots, 2}^{32}, \ldots]$$

and similar numbers are transcendental; see [16]. Previously, Maillet had given similar examples, but not explicitly [201]. Other examples have been recently given by Davison [79]. Also see Grant [120].

10. "Quasi-Monte-Carlo" Methods and Zaremba's Conjecture

In this section we briefly discuss some integration methods that depend on rational numbers with small partial quotients. There is a large literature on this subject; the interested reader can start with the comprehensive survey of Niederreiter [220].

(This section is tied to the main subject in the following manner: we wish to construct explicitly rational numbers with small partial quotients. One way to do this is to take an irrational number with bounded partial quotients and employ the sequence of convergents.)

In s-dimensional "quasi-Monte Carlo" integration, we approximate the integral

$$\int_{[0,1]^s} f(\mathbf{t})\,d\mathbf{t} \tag{2}$$

by the sum

$$\frac{1}{n} \sum_{1 \leqslant k \leqslant n} f(\mathbf{x}_k),$$

where $\mathbf{x}_1, \mathbf{x}_2, \ldots$ is a set of points in $[0, 1]^s$.

The goal of quasi-Monte Carlo integration is to choose the points $\mathbf{x}_1, \mathbf{x}_2, \ldots$ so as to minimize the error in the approximation.

In the method of *good lattice points,* we assume that the function f is periodic of period 1 in each variable. We choose a large fixed integer m and a special lattice point $\mathbf{g} \in \mathbf{Z}^s$. Then we approximate the integral (2) with the sum

$$\frac{1}{m} \sum_{1 \leqslant k \leqslant m} f\left(\frac{k}{m}\mathbf{g}\right).$$

"Good" lattice points $\mathbf{g}$ make the error in this approximation relatively small.

Let $\mathbf{h} = (h_1, h_2, \ldots, h_s)$ and define

$$r(\mathbf{h}) = \prod_{1 \leqslant i \leqslant s} \max(1, |h_i|) .$$

Also define

$$\rho(\mathbf{g}, m) = \min_{\mathbf{h}} r(\mathbf{h}) ,$$

where $\mathbf{h}$ ranges over all lattice points with

$$\frac{-m}{2} < h_j \leqslant \frac{m}{2} ,$$

$\mathbf{h} \neq 0$, and $\mathbf{h} \cdot \mathbf{g} \equiv 0 \pmod{m}$. It can be shown that good lattice points correspond to large values of ρ.

Now consider the 2-dimensional case, i.e. $s = 2$. Let $\mathbf{g} = (1, g)$ with $\gcd(g, m) = 1$. Then Zaremba [306] showed that

$$\frac{m}{K(g/m) + 2} \leqslant \rho(\mathbf{g}, m) \leqslant \frac{m}{K(g/m)} .$$

Hence good lattice points correspond to rationals g/m with small partial quotients.

For other connections with numerical integration, see the papers of Haber and Osgood [125] and Zaremba [307].

We now turn to Zaremba's conjecture. Define

$$Z(n) = \min_{\substack{1 \leqslant j \leqslant n \\ \gcd(j, n) = 1}} K\left(\frac{j}{n}\right) .$$

Then Zaremba [307] conjectured that $Z(n) \leqslant 5$.

Borosh [41] showed that Zaremba's conjecture is true for $1 \leqslant n \leqslant 10000$. In this range, only two integers have $Z(n) = 5$, namely $n = 54$ and $n = 150$. Twenty-five integers in this range have $Z(n) = 4$; the smallest is 6 and the largest is 6234. A brief discussion of Zaremba's conjecture up to 1978 can be found in [220].

Borosh and Niederreiter [42] suggested that in fact $Z(n) \leqslant 3$ for all sufficiently large n. The most extensive computation seems to be that of Knuth, cited in [42], which verified that $Z(n) \leqslant 3$ for $10000 \leqslant n \leqslant 3200000$.

Zaremba's conjecture is true for certain infinite sequences. For example, we certainly have $Z(F_k) = 1$ for $k \geqslant 1$, where F_k is the k-th Fibonacci number. It follows from the results of Kmošek and Shallit cited above that $Z(2^{2^k-1}) \leqslant 2$ for all $k \geqslant 0$.

Borosh and Niederreiter [42] showed that $Z(2^k) \leqslant 3$ for $6 \leqslant k \leqslant 35$.

More recently, Niederreiter [223] proved that Zaremba's conjecture holds for all powers of 2; in fact, we have $Z(2^k) \leqslant 3$ for all $k \geqslant 0$.

Larcher [182, Corollary 2] proved the existence of a constant c, such that for every $n \geqslant 1$ there exists a positive integer $j \leqslant n$, relatively prime to n, such that if

$$j/n = [0, a_1, a_2, \ldots, a_m] \ ,$$

then

$$\sum_{1 \leqslant i \leqslant m} a_i < c(\log n)\,(\log \log n)^2 \ .$$

This is close to the best possible bound $O(\log n)$, which was reportedly conjectured by L. Moser (although I do not know a reference); the bound would be a consequence of Zaremba's conjecture.

For other results connected with Zaremba's conjecture, see the papers of Cusick [63, 66]; Niederreiter [224]; Sander [268]; and Hensley [315].

11. Properties of the sequence $n\theta$ (mod 1)

If θ is a real number, by θ (mod 1) we mean $\{\theta\} = \theta - [\theta]$, the fractional part of θ.

It has been known at least since Bernoulli [26] that properties of the sequence $\theta, 2\theta, 3\theta, \ldots$ are intimately connected with the continued fraction expansion for θ. The distribution of $n\theta$ (mod 1) is a vast subject, and we restrict ourselves to mentioning several results connected with numbers of constant type.

Let θ be an irrational number, and let

$$0 = a_0 < a_1 < a_2 < \cdots < a_n < a_{n+1} = 1$$

be the sequence of points $\{k\theta\}$, $1 \leqslant k \leqslant n$, arranged in ascending order. Define

$$\delta_\theta(n) = \max_{1 \leqslant i \leqslant n+1} a_i - a_{i-1} \ .$$