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of C belongs to the interior of the polynomial hull of \bar{W} , then C is entirely included in the interior of the polynomial hull of \bar{W} .

By holomorphic curve we will mean a connected 1-dimensional holomorphic manifold.

Proof. Let O be the interior of the polynomial hull of \bar{W} . It has to be shown that the set of points $p \in C$ which belong to O is closed in C . It is obviously open. Things being so localized one has to face the following situation: a "small" analytic disk given by a holomorphic parametrization $\varphi: \bar{\Delta} \rightarrow C$ (Δ the unit disk in \mathbf{C}) so that $\varphi(1) \in O$, U^+ a side of M included in W (at least one of the two sides is such) hence in O , in some neighborhood of $\varphi(\bar{\Delta})$; and one has to show that $\varphi(0) \in O$. Fix ψ a holomorphic map from \mathbf{C} into \mathbf{C}^n so that: $\psi(e^{i\theta}) \simeq -\vec{N}$ for θ outside some small neighborhood of $0 \pmod{2\pi}$, where \vec{N} is the unit outer normal to M (with respect to U^+), at say the point $\varphi(0)$, and $\psi(0)$ is arbitrarily chosen.

For $\eta > 0$, η small enough $\varphi(e^{i\theta}) + \eta\psi(e^{i\theta}) \in O$ for all θ , hence $\varphi(0) + \eta\psi(0) \in O$. Taking into account some uniformity with respect to $\psi(0)$, this gives Lemma 2.

III. LIFTING TO \mathbf{C}^3

We are simply going to consider sets K in \mathbf{C}^3 rotationally invariant in the first variable, that we describe as follows. For each $t \in [0, t_0]$ we are given a compact set $K_t \subset \mathbf{C}^2$. We consider the set $K \subset \mathbf{C}^3$ which is the closure of the set $\{(w, z_1, z_2) \in \mathbf{C}^3; (z_1, z_2) \in K_{|w|}, |w| \leq t_0\}$. i.e.

$$K = \overline{\bigcup_{|w| \leq t_0} \{w\} \times K_{|w|}}.$$

\hat{K} denotes the polynomial hull of K in \mathbf{C}^3 , while $\hat{\bigcup}_{t \leq t_0} K_t$ denotes the polynomial hull in \mathbf{C}^2 of the closure of the set $\bigcup_{t \leq t_0} K_t$.

LEMMA 3. Let $(0, \zeta_1, \zeta_2) \in \mathbf{C}^3$, the following are equivalent:

$$\begin{cases} (i) & (0, \zeta_1, \zeta_2) \in \hat{K} \\ (ii) & (\zeta_1, \zeta_2) \in \hat{\bigcup}_{t \leq t_0} K_t. \end{cases}$$

Proof. (i) \Rightarrow (ii) is trivial. We are interested in (ii) \Rightarrow (i). Let $P(w, z_1, z_2)$ be a polynomial in 3 variables. To P we associate the polynomial \tilde{P} defined by

$$\tilde{P}(w, z_1, z_2) = P(0, z_1, z_2) = \frac{1}{2\pi} \int_0^{2\pi} P(e^{i\theta} w, z_1, z_2) d\theta.$$

Since K is invariant under rotation in the w variable:

$$\sup_K |\tilde{P}| \leq \sup_K |P|.$$

Set $Q(z_1, z_2) = P(0, z_1, z_2)$. Using (ii) one gets

$$|P(0, \zeta_1, \zeta_2)| = |Q(\zeta_1, \zeta_2)| \leq \sup_{\cup K_t} |Q| = \sup_K |\tilde{P}| \leq \sup_K |P|.$$

So (i) is established.

Remark. There is another approach to Lemma 3, which may better “explain” the situation, and that we just sketch. If $\varphi: \Delta \rightarrow \mathbf{C}^2$ is a holomorphic disk (φ continuous on $\bar{\Delta}$, holomorphic on Δ) and T is a continuous map from $\mathbf{R}/2\pi\mathbf{Z}$ into $[0, t_0]$ so that $\varphi(e^{i\theta}) \in K_{T(\theta)}$ ($\theta \in [0, 2\pi)$), then $\varphi(0) \in \hat{\cup} K_t$. One sees that $(0, \varphi(0)) \in \hat{K}$ by considering holomorphic disks $(Q, \varphi): \Delta \rightarrow \mathbf{C} \times \mathbf{C}^2$, with $Q(0) = 0$ and $|Q(e^{i\theta})| \simeq T(\theta)$. Carrying this out in general may require the use of the fundamental theorem by Poletsky [6], which says that, in an appropriate sense, polynomial hulls are always explained by holomorphic disks.

IV. TREPEAU'S EXAMPLE

Here we describe a class of examples. Let χ be a smooth real valued function defined on $[0, 1]$, constant in no neighborhood of 0, and so that $\chi(0) = 0$, $|\chi| < 1$. In one of the versions of Trepreau's original example $\chi(t) = t$. Let \mathcal{M} be the generic 4-dimensional manifold in \mathbf{C}^3 , given by:

$$\begin{aligned} \mathcal{M} = \{ & (w, z_1, z_2) \in \mathbf{C}^3, |w| < 1, z_1 = s_1 + i\chi(|w|^2)s_2, \\ & z_2 = s_2 - i\chi(|w|^2)s_1; (s_1, s_2) \in \mathbf{R}^2 \}. \end{aligned}$$

Notice that on \mathcal{M} , $z_1^2 + z_2^2$ is a real valued function, (on \mathcal{M} , $z_1^2 + z_2^2 \geq 0$), hence:

(*) Any function which depends only on $(z_1^2 + z_2^2)$ is a CR function on \mathcal{M} .

This already gives example of CR functions which cannot be holomorphically extended to any wedge. The existence of such functions is related to the fact that \mathcal{M} is not “minimal” (in the sense of Tumanov), it contains $\mathbf{C} \times \{0\} \times \{0\}$ as a (nongeneric) CR manifold of same CR dimension (see [9], [2]).