

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 39 (1993)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: HURWITZ QUATERNIONIC INTEGERS AND SEIFERT FORMS
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Kapitel: §4. Automorphisms of the root System $\$nD_4\$$ and perfect isometries
DOI: <https://doi.org/10.5169/seals-60415>

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Proof. Note that every perfect isometry σ of \mathcal{H} extends naturally to a perfect isometry of \mathcal{H}^* , inducing a perfect \mathbf{F}_2 -isomorphism $\eta(\sigma)$ of $\mathcal{H}^*/\mathcal{H}$, η denoting the induced map on the quotient. The proof of the proposition is complete in view of the following simple lemma.

3.5. LEMMA. *An \mathbf{F}_2 -linear isomorphism of \mathbf{F}_4 is perfect if and only if it corresponds to multiplication by ω , where ω denotes a primitive element of \mathbf{F}_4 over \mathbf{F}_2 .*

Proof. An \mathbf{F}_2 -linear isomorphism of \mathbf{F}_4 is perfect if and only if it has no fixed point other than the trivial element. Since, $GL_2(\mathbf{F}_2) \simeq S_3$, it is easy to see that every perfect isomorphism of \mathbf{F}_4 , corresponds to multiplication by ω , ω being as above.

3.6. PROPOSITION. *Let L be a \mathbf{Z} -lattice such that $\mathcal{H}^n \subseteq L \subseteq \mathcal{H}^{*n}$. If L is an \mathcal{H} -lattice, then L has a perfect isometry, which corresponds to multiplication by ω , on the quotient $\mathcal{H}^{*n}/\mathcal{H}^n$.*

Proof. Multiplication by ξ is a perfect isometry of \mathcal{H}^n which extends naturally to a perfect isometry of \mathcal{H}^{*n} . Clearly the induced map on the quotient $\mathcal{H}^{*n}/\mathcal{H}^n$ is multiplication by ω . Since L is an \mathcal{H} -module, it preserves L as well.

In particular,

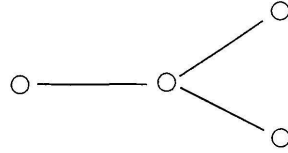
3.7. COROLLARY. *Every \mathcal{H} -lattice $(L, Tr \circ h)$ of type nD_4 has a perfect isometry.*

It is but natural to ask whether every \mathbf{Z} -lattice of type nD_4 which has a perfect isometry necessarily admits the structure of an \mathcal{H} -lattice. We shall show that this is indeed true. For doing this we need to recall some basic facts on the automorphisms of the root system nD_4 .

§4. AUTOMORPHISMS OF THE ROOT SYSTEM nD_4 AND PERFECT ISOMETRIES

For any root system R , let $\mathcal{W}(R)$ denote the Weyl group of R (i.e. the group generated by the reflections defined by the roots). Then $\mathcal{W}(R)$ is a normal subgroup of $Aut R$, which preserves every \mathbf{Z} -lattice L such that $\mathbf{Z}R \subseteq L \subseteq \mathbf{Z}R^\#$. We thus get a natural map $\eta: Aut R/\mathcal{W}(R) \rightarrow Aut_{\mathbf{Z}}(\mathbf{Z}R^\#/\mathbf{Z}R)$. In view of ([H], p. 72; [C-S], p. 432) this is an injection.

An element σ in $Aut(\mathbf{R})/\mathcal{W}(\mathbf{R})$ preserves L if and only if $\eta(\sigma)$ preserves the corresponding subgroup $\eta(L)$ of $\mathbf{Z}\mathbf{R}^\#/\mathbf{Z}\mathbf{R}$. If $\mathbf{R} = \mathbf{D}_4$, $Aut \mathbf{R} = \mathcal{W}(\mathbf{R}) \times_s S_3$, where, \times_s denotes the semi direct product and S_3 is the automorphism group of the associated Dynkin diagram:



Consequently, for $\mathbf{R} = n\mathbf{D}_4$, $Aut \mathbf{R}/\mathcal{W}(\mathbf{R}) \simeq S_3^n \times_s S_n \simeq (GL_2(\mathbf{F}_2))^n \times_s S_n$. Thus the elements of $Aut \mathbf{R}/\mathcal{W}(\mathbf{R})$ are "monomial matrices" where each row and each column consists of exactly one element of $GL_2(\mathbf{F}_2)$. It acts naturally on $(\mathbf{Z}\mathbf{D}_4^\#)^n/\mathbf{Z}\mathbf{D}_4^n$. In view of the identification of $\mathbf{Z}\mathbf{D}_4^\#/\mathbf{Z}\mathbf{D}_4 \simeq \mathcal{H}^*/\mathcal{H}$, we have the following proposition.

4.1. PROPOSITION.

- (a) $Aut(\mathcal{H}^n)/\mathcal{W}(\mathcal{H}^n) \simeq S_3^n \times_s S_n \simeq (GL_2(\mathbf{F}_2))^n \times_s S_n$.
- (b) If U denotes the group of units of \mathcal{H} , then U is a subgroup of $Aut \mathcal{H}$ and $U/(\mathcal{W}(\mathcal{H}) \cap U) \simeq \{1, \omega, \omega^2\}$, where $\mathbf{F}_2(\omega) = \mathbf{F}_4$.
- (c) The conjugation in \mathcal{H} belongs to the Weyl group $\mathcal{W}(\mathcal{H})$.

Proof. (a) This statement is an immediate consequence of the identification $\mathbf{Z}\mathbf{D}_4 \simeq \mathcal{H}$.

(b) By (a), $Aut \mathcal{H}/\mathcal{W}(\mathcal{H}) \simeq S_3 \simeq GL_2(\mathbf{F}_2)$. Since $\eta(U) = \{1, \omega, \omega^2\}$, (b) follows.

(c) The conjugation in \mathcal{H} is a product of reflections defined by i, j and k .

We now consider the perfect isomorphisms of $(\mathcal{H}^{*n})/\mathcal{H}^n$ arising out of $Aut(\mathcal{H}^n)/\mathcal{W}(\mathcal{H}^n)$. We begin by fixing the following notation:

Let $V = \mathbf{F}_4^n = X_1 \perp X_2 \perp \dots \perp X_n$ with respect to the standard hermitian form on V , where $X_i \simeq \mathbf{F}_4 = \mathbf{F}_2 \oplus \mathbf{F}_2 = \{0, 1, \omega, \omega^2\}$. Let G denote the group of all $n \times n$ monomial matrices with entries in $M_2(\mathbf{F}_2)$, where each row and each column consists of exactly one element of $GL_2(\mathbf{F}_2)$. Note that every element of G can be uniquely expressed as $\alpha \cdot \tau$, where α is the diagonal matrix $\text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$, with α_i in $GL_2(\mathbf{F}_2)$ and τ is an $n \times n$ permutation matrix. We have,

4.2. LEMMA. Let σ belonging to G be perfect and let $X = X_i$ for some i . Let m be the smallest positive integer for which σ^m maps X onto itself. Then σ^m/X is perfect.

Proof. The idea of the proof is similar to ([K], Prop. 2). We show that $(1 - \sigma^m)/X$ is surjective. Let $M = \sum_{0 \leq i \leq m-1} \sigma^i(X)$. Then σ leaves M invariant. Therefore σ is a perfect isomorphism of M . Hence $(1 - \sigma)/M: M \rightarrow M$ is surjective. Let x be an element of X . Since, $(x, 0, \dots, 0)$ belongs to M , there exists an element y in M such that $(1 - \sigma)(y) = (x, 0, \dots, 0)$. Let $y = (y_0, y_1, \dots, y_{m-1})$, where y_i belongs to $\sigma^i(X)$. Then,

$$(1 - \sigma)(y) = (y_0 - \sigma(y_{m-1}), y_1 - \sigma(y_0), \dots, y_{m-1} - \sigma(y_{m-2})).$$

Hence, $y_0 - \sigma(y_{m-1}) = x, y_1 = \sigma(y_0), \dots, y_{m-1} = \sigma(y_{m-2})$. Further, $\sigma(y_{m-1}) = \sigma^2(y_{m-2}) = \dots = \sigma^m(y_0)$. Thus $(1 - \sigma^m)(y_0) = x$. This implies that $(1 - \sigma^m)/X$ is surjective.

4.3. COROLLARY. Let σ be an element of G which is perfect. Suppose that $\sigma = \alpha \cdot \tau$, where $\alpha = \text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$, $\alpha_i \in GL_2(\mathbb{F}_2)$, $\tau = \tau_1 \cdot \tau_2 \dots \tau_r$, and τ_i are disjoint cyclic permutations of length n_i . Let T_i denote the set of indices belonging to the permutation τ_i . Then $(\sigma)^{n_i}/X_j$ is perfect for every j belonging to T_i .

Proof. Note that for every j belonging to T_i , n_i is the smallest positive integer such that $(\sigma)^{n_i}$ maps X_j onto itself.

4.4. COROLLARY. If σ is as above, then $(\sigma)^{n_i}/X_j$ corresponds to multiplication by ω or ω^2 , for every j belonging to T_i .

Proof. Follows from Corollary 4.3, and Lemma 3.5.

4.5. COROLLARY. If σ is as above, and $X^{(i)} = \sum_{j \in T_i} X_j$, then $(\sigma)^{n_i}/X^{(i)}$ is the matrix $\text{diag}(\alpha_1, \dots, \alpha_j, \dots, \alpha_{n_i})$, where α_j belongs to $\{\omega, \omega^2\}$.

Proof. Clear from Corollary 4.4.

4.6. PROPOSITION. Let σ be an element of G which is perfect and let $\sigma = \alpha \cdot \tau$, where α and τ are as in Corollary 4.4. Then there exists an integer $l \geq 1$, such that σ^l is perfect and $\sigma^l = \beta \cdot \tau'$, where β is the matrix $\text{diag}(\beta_1, \dots, \beta_j, \dots, \beta_n)$, with β_j in $GL_2(\mathbb{F}_2)$ and τ' is a product of disjoint cyclic permutations τ_i of length 3^{k_i} .

Proof. Let $\tau = \tau_1 \cdot \tau_2 \dots \tau_r$, where τ_i are disjoint cyclic permutations of length $n_i = 3^{k_i} \cdot l_i$, with $(3, l_i) = 1$. Let l denote the least common multiple of the l_i . We show that σ^l is perfect. By Corollary 4.5, σ^{n_i}/X_j is multiplication by ω or ω^2 for every j belonging to T_i . This implies that $(\sigma)^{n_i l/l_i}/X_j$ corresponds to multiplication by ω or ω^2 for every such j , since $(l/l_i, 3) = 1$ and ω is an element of order 3. Hence, $(\sigma^l)^{3^{k_i}}/X^{(i)}$ is the matrix $\text{diag}(\alpha_1, \dots, \alpha_j, \dots, \alpha_{n_i})$ where α_j belongs to $\{\omega, \omega^2\}$. Clearly this implies that $\sigma^l/X^{(i)}$ has no nontrivial fixed point. Since T_i are disjoint, it follows that σ^l has no nontrivial fixed point and hence σ^l is perfect. Obviously σ^l has the required property and the proposition follows.

Now, let M be an \mathbf{F}_2 -linear subspace of V , which is invariant under a perfect isomorphism σ belonging to G . By the previous proposition, we can assume, by replacing σ by σ^m , that M is invariant under $\sigma = \alpha \cdot \tau$, where α is as in Corollary 4.4 and $\tau = \tau_1 \cdot \tau_2 \dots \tau_r$, τ_i being cyclic permutations of length 3^{k_i} .

4.7. PROPOSITION. *If M is an \mathbf{F}_2 -linear subspace of V which has a perfect isomorphism σ belonging to G , then M is invariant under the action of a diagonal matrix, $\text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$ where each α_i belongs to $\{\omega, \omega^2\}$.*

Proof. By replacing σ by a suitable power we may assume that

$$\sigma = \text{diag}(\beta_1, \dots, \beta_i, \dots, \beta_n) \tau_1 \tau_2 \dots \tau_r$$

where β_i belongs to $GL_2(\mathbf{F}_2)$ for every i and τ_i are disjoint cyclic permutations of length 3^{k_i} . Further, since disjoint cycles commute we may assume that the length of τ_i is 3^k for $1 \leq i \leq s$ and the length of τ_i is less than 3^k for $s < i \leq r$. Let $T = \{i \in \{1, 2, \dots, n\} \mid i \text{ occurs in the permutation } \tau_1 \tau_2 \dots \tau_s\}$. Let $M_1 = M \cap \sum_{i \in T} X_i$ and $N_1 = M \cap \sum_{i \notin T} X_i$. We claim that $M = M_1 \oplus N_1$

and that M_1 is invariant under $\text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$, where each α_i belongs to $\{\omega, \omega^2\}$. Let $(x, y) \in M$, where $x \in \perp_{i \in T} X_i$, $y \in \perp_{i \notin T} X_i$. Since

$$\sigma^{3^k} = \text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n),$$

where α_i belongs to $\{\omega, \omega^2\}$ for $i \in T$ and $\alpha_i = 1$ for $i \notin T$, it follows that, $(x, y) + \sigma^{3^k}(x, y) + (\sigma^{3^k})^2(x, y) = (0, y)$ belongs to M . Hence $(x, 0)$ belongs to M as well. Thus $M = M_1 \oplus N_1$. Clearly M_1 is invariant under $\text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$, α_i being in $\{\omega, \omega^2\}$. Since σ/N_1 is perfect, by

repeating the above argument we obtain a similar decomposition of $N_1: N_1 = M_2 \oplus N_2$. This process terminates in a finite number of steps and we obtain a decomposition $M = M_1 \oplus M_2 \oplus \dots \oplus M_k$, where each M_j is invariant under $\text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$, α_i being in $\{\omega, \omega^2\}$.

§5. MAIN THEOREM AND EXAMPLES

In this final section we prove our main results 5.2, 5.3 and give some examples. We begin with,

5.1. PROPOSITION. *Let L be a unimodular \mathbf{Z} -lattice of type nD_4 such that $\mathcal{H}^n \subset L \subset \mathcal{H}^{*n}$. If L admits a perfect isometry, then there exists an isometry $\delta = \text{diag}(\delta_1, \dots, \delta_i, \dots, \delta_n)$ on \mathcal{H}^{*n} , where δ_i is the isometry on \mathcal{H}^* given by left multiplication by ξ or right multiplication by $\bar{\xi}$ such that L is invariant under δ .*

Proof. Let σ be a perfect isometry of $(L, \text{Tr} \circ h)$. Then σ induces an automorphism of \mathcal{H}^n and extends naturally to a perfect isometry of \mathcal{H}^{*n} . In view of ([K], p. 179), $\eta(\sigma)$ is a perfect isomorphism of \mathbf{F}_4^n , leaving $\eta(L)$ invariant. Therefore by Proposition 4.7 there exists $\alpha = \text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$ with α_i in $\{\omega, \omega^2\}$ such that $\eta(L)$ is invariant under α . Let δ_i denote left multiplication on \mathcal{H}^* by $\xi = (1 + i + j + k)/2$ if $\alpha_i = \omega$ and right multiplication by $\bar{\xi} = (1 - i - j - k)/2$, if $\alpha_i = \omega^2$. Let $\delta = \text{diag}(\delta_1, \dots, \delta_i, \dots, \delta_n)$. Since δ induces an isometry of \mathcal{H}^{*n} which fixes \mathcal{H}^n and $\eta(\delta) = \alpha$ leaves $\eta(L)$ invariant it follows that δ leaves L invariant.

5.2. THEOREM. *Let (L, S) be an unimodular \mathbf{Z} -lattice of type nD_4 . Then, L has a perfect isometry if and only if there exists an \mathcal{H} -lattice (L', S') such that $L \simeq L'$.*

Proof. Clearly every \mathcal{H} -lattice admits a perfect isometry (3.2). Conversely let (L, S) be a \mathbf{Z} -lattice of type nD_4 , which admits a perfect isometry. In view of Proposition 2.1, we can assume that $\mathcal{H}^n \subseteq L \subseteq \mathcal{H}^{*n}$ and $S = \text{Tr} \circ h$. By Proposition 4.7 there exists a subset T of $\{1, 2, \dots, n\}$ such that L is invariant under $\delta = (\delta_1, \dots, \delta_i, \dots, \delta_n)$, where δ_i is left multiplication by ξ for $i \in T$ and δ_i is right multiplication by $\bar{\xi}$ for $i \notin T$. Let $f: \mathcal{H}^n \rightarrow \mathcal{H}^n$ be defined by $f = \text{diag}(f_1, \dots, f_i, \dots, f_n)$ where $f_i = \text{id}$ for $i \in T$ and $f_i =$ the involution on \mathcal{H} for $i \notin T$. Then it is easy to check that f is an isometry of $(L, \text{Tr} \circ h)$ onto (L', S') where, $L' = f(L)$, and,

$$S'(x, y) = \sum_{i \in T} (x_i \bar{y}_i + y_i \bar{x}_i) + \sum_{i \notin T} (\bar{x}_i y_i + \bar{y}_i x_i).$$