Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 40 (1994)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: PROOF OF MARGULIS' THEOREM ON VALUES OF QUADRATIC FORMS, INDEPENDENT OF THE AXIOM OF CHOICE

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Anhang: Appendix: Recurrent points

DOI: https://doi.org/10.5169/seals-61104

V_1-invariant subsets of X and $v_2(t_k)Y_k \subseteq X$ for all k. Also it is easy to see that $\{t_k \mid k \geqslant 1\}$ contains either all positive rational or all negative rational numbers. Now let $Y' = \cap_{k=1}^{\infty} Y_k$. Since $\{Y_k\}$ is a decreasing sequence of compact subsets, Y' is nonempty. Now if $\{t_k \mid k \geqslant 1\}$ contains all positive rational numbers then $v_2(r)Y' \subseteq X$ for all positive rational numbers r and hence by continuity $V_2^+ Y' \subseteq X$ and, similarly, in the alternative case $V_2^- Y' \subseteq X$. This completes the proof of the theorem.

Appendix: Recurrent points

For a compact metric space X we denote by $C(X)$ the space of all continuous real-valued functions on X equipped with the sup-norm topology and by $C(X)^+$ the subset of $C(X)$ consisting of all nonnegative functions; the supremum norm of $f \in C(X)$, namely $\sup\{\,|f(x)|\;|x \in X\}$, will be denoted by $\|f\|$. By an integral on $C(X)$ we mean a linear functional on $C(X)$ which takes nonnegative values on $C(X)^+$. For an integral Λ on $C(X)$ the *support* of Λ is defined to be the subset of X consisting of all $x \in X$ such that $\Lambda(f) > 0$ for any $f \in C(X)^+$ for which $f(x) > 0$; the support is easily seen to be a closed subset of X. It can also be verified by a simple point-set topological argument that if Λ is an integral on $C(X)$ and $f \in C(X)$ vanishes on the support of Λ then $\Lambda(f) = 0$. If Λ is an integral on $C(X)$, where X is a compact metrizable space, and X' is the support of Λ then there exists a unique integral Λ' on $C(X')$ such that $\Lambda'(f|_{X'}) = \Lambda(f)$ for all $f \in C(X)$, where $f|_{X'}$ denotes the restriction of f to X'; this follows from the Tietze-Urysohn extension theorem (cf. [D], (4.5.1)) and the above mentioned property of the support. We note also that the support of Λ' as above is the whole of X'.

For any homeomorphism φ of a compact (metrizable) space X an integral Λ on $C(X)$ is said to be φ-invariant if $\Lambda(f \circ \varphi) = \Lambda(f)$ for all $f \in C(X)$; clearly the support of a φ-invariant integral on $C(X)$ is a φ-invariant (closed) subset of X.

Proof of Proposition 1.7. We fix a dense sequence in $C(X)$, say f_j, $j = 1, 2, \ldots$. Let $x_0 \in X$. Given f_j, for any sequence $\{m_k\}$ of natural numbers $m_k^{-1} \sum_{i=0}^{m_k - 1} f_j \circ \varphi^i(x_0)$ is a bounded sequence and therefore admits a convergent subsequence. Using a standard procedure (finding $\{m_k^{(j)}\}$, with each sequence a subsequence of the previous one, such that the corresponding sequence for f_j as above converges and considering $\{m_k^{(k)}\}$) we get a sequence $\{n_k\}$ of natural numbers such that $n_k^{-1} \sum_{i=0}^{n_k - 1} f_j \circ \varphi^i(x_0)$ converges for all j; also, the limit is between $-\|f_j\|$ and $\|f_j\|$. Since $\{f_j\}$ is dense in $C(X)$ this readily implies that $n_k^{-1} \sum_{i=0}^{n_k - 1} f \circ \varphi^i(x_0)$ converges for all $f \in C(X)$; let c_f be the limit corresponding to f. Then it can be verified that $\Lambda: C(X) \to \mathbf{R}$ defined by $\Lambda(f) = c_f$, for all $f \in C(X)$, is a φ-invariant integral on $C(X)$. Also clearly Λ is not identically zero and therefore by our observations above, the support, say X', is a nonempty closed φ-invariant subset of X and further $C(X')$ admits an integral with full support (namely X') which is invariant under the restriction of φ to X'. Replacing X as in the hypothesis by X' we may without loss of generality assume that $C(X)$ admits a φ-invariant integral whose support is X; in the rest of the argument we let Λ be any such integral.

Now suppose that there do not exist any recurrent points for φ. Let $\rho(\cdot, \cdot)$ be the metric on X. Let θ be the function on X defined by $\theta(x) = \inf\{\rho(\varphi^i(x), x) \mid i = 1, 2, \ldots\}$, for all $x \in X$. There being no recurrent points means that $\theta(x) > 0$ for all $x \in X$. For each natural number k let $E_k = \{x \in X \mid \theta(x) \geqslant 1/k\}$. Then each E_k is a closed subset of X and $X = \cup E_k$. Therefore by the Baire category theorem there exists a k such that E_k has an interior point in X. In particular, there exists an open ball, say A, of radius at most $1/3k$ contained in E_k. The definition of E_k and the condition on the radius of A then imply that the sets $\varphi^i(A)$, $i \in \mathbf{Z}$, are mutually disjoint. Now let $x \in A$ and let $f \in C(X)^+$ be such that $f(x) > 0$ and the support of f (the closure of the set $\{y \in X \mid f(y) > 0\}$) is contained in A. For each natural number n let $S_n(f) = \sum_{i=0}^{n-1} f \circ \varphi^i \in C(X)$. The disjointness of $\varphi^i(A)$, $i \in \mathbf{Z}$, implies that, for any n, $\| S_n(f) \| = \| f \|$. Also, by the φ-invariance of Λ we have $\Lambda(S_n(f)) = n\Lambda(f)$. Hence $\Lambda(f) = \Lambda(S_n(f))/n \leqslant \| S_n(f) \| \Lambda(1_X)/n = \| f \| \Lambda(1_X)/n$ for all n, where 1_X denotes the constant function with value 1. But this implies that $\Lambda(f) = 0$ contradicting the assumption that the support of Λ is the whole of X. This proves the proposition.

Acknowledgement: Thanks are due to J. Aaronson and M.G. Nadkarni for a discussion on recurrent points and to Gopal Prasad and Nimish Shah for useful comments on an earlier version.

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