Introduction

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HIGHER EULER CHARACTERISTICS (I)

by Ross GEOGHEGAN[1]) and Andrew NICAS[2])

To Peter Hilton on the occasion of his 70-th birthday.

ABSTRACT. The classical Euler characteristic $\chi \equiv \chi_0$ of a finite complex lies at the bottom of a sequence of homotopy invariants. The next invariant in this sequence χ_1 is introduced here and studied in some detail. The rest of the sequence, χ_n with $n \geqslant 2$, will be discussed in a sequel paper. Applications to geometric group theory are found by considering the behavior of χ_1 on an aspherical finite complex of fundamental group G. Just as the $\chi(G) \neq 0$ implies that the center of G is trivial (Gottlieb's Theorem), it is shown here that (under a weak additional hypothesis and using rational coefficients) $\chi_1(G) \neq 0$ implies that the center of G is infinite cyclic. We also find a generalization of Gottlieb's Theorem in which the Lefschetz number of an automorphism of G is related to the fixed subgroup of the automorphism.

INTRODUCTION

From our point of view, the classical Euler characteristic of a finite complex is "zero-th order". In this paper we introduce a "first order" analog, a new invariant in topology and group theory. In a sequel paper and in [GNO] we extend these ideas to an "n-th order" Euler characteristic for all positive n.

For a finite complex X, the new invariant $\chi_1(X; R)$, defined in § 1, comes in different forms, depending on the coefficient ring R; and a more sophisticated version $\tilde{\chi}_1(X; R)$ defined in § 2, involves the universal cover of X. By contrast, the classical analogs of these are essentially the same, namely the integer $\chi(X)$. We should tell the reader from the start that all our first order invariants are trivial if X is simply connected.

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The paper begins with three rather different definitions of $\chi_1(X; R)$, a discussion of their equivalence, and some motivation for these definitions. Our point of view is geometric, but for readers more interested in homotopy theory we include (at the end of § 1) a brief discussion of a fourth definition in terms of stable homotopy theory.

Next, we discuss the computation of $\chi_1(X; R)$ for 1-complexes, certain 2-complexes, 3-dimensional lens spaces, circle bundles and mapping tori.

In § 5 and § 7, we apply these ideas to group theory. Motivated by Gottlieb's theorem [Got] that if X is a finite aspherical complex with fundamental group G and if $\chi(G) \equiv \chi(X) \neq 0$ then the center of G is trivial, we find an analog (Theorem 5.4) which says, roughly, that if $\chi_1(G; \mathbf{Q}) \equiv \chi_1(X; \mathbf{Q}) \neq 0$ then the center of G is infinite cyclic. This leads us to surprising generalization of Gottlieb's theorem (Theorem 8.1). In this theorem, one is given an automorphism θ of G induced by a map $f: X \to X$. By the *Lefschetz number,* $L(\theta)$, of θ we mean the Lefschetz number of f. We prove (under a weak K-theoretic hypothesis on G) that if θ has order r in the group of outer automorphisms of G and if $\sum_{i=0}^{r-1} L(\theta^i) \neq 0$ then the intersection of the center of G and the fixed subgroup of θ is trivial. We do not know of a previous theorem which relates so directly the fixed subgroup of an automorphism to the classical fixed point theory of the associated map.

We also introduce a more refined invariant $\tilde{\chi}_1(X) \in H^1(\Gamma, HH_1(\mathbf{Z}G))$ where $HH_1(\mathbf{Z}G)$ is the first Hochschild homology group of $\mathbf{Z}G$ (see § 1). This is an analog of what one obtains when one computes the classical Euler characteristic as a Hattori-Stallings trace in the universal cover of X. In the classical case one essentially recovers $\chi(X)$, but a significant distinction appears in the case of the "higher order" invariants. In a natural manner, $\tilde{\chi}_1(X)$ maps to $\chi_1(X)$ regarded as an element of $H^1(\Gamma, H_1(G))$. Applications of $\tilde{\chi}_1$ to characteristic classes and Seifert fiber spaces will be given in [GN_5].

The ideas presented here are an outgrowth of the one-parameter fixed point theory developed in [GN_1] and its application to dynamics in [GN_2]. A summary has appeared as [GN_3]. Most of this paper can be read independently of [GN_1] and [GN_2]. We make modest use of a few technical propositions from [GN_1] in § 2 and § 3; in § 10 a difficult result from [GN_1] is invoked.

One of the definitions of $\chi_1(X; R)$ employs a formula introduced more than twenty years ago in [Kn]; we thank Boris Okun for drawing our attention to that paper.