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homology class in F , up to sign. It follows, therefore, from the classification of surfaces, that each X_ℓ has Euler characteristic ≤ -1 . Therefore, when $n > 1$ and $n = g$,

$$\chi(F) = \chi(\bar{F}) = 2 - 2n = \sum \chi(X_\ell) \leq (\text{card } S - n + 1)(-1)$$

or equivalently $\text{card } S \leq 3n - 3$, as required.

It remains to consider the case when $g > n > 1$. In this case, we first proceed as before, cutting open along the A_i , obtaining a connected surface \hat{F} of genus $g - n$ and with $2n$ boundary curves, containing the m curves C_j , each of which is homologous to a sum of boundary curves in \hat{F} . Now each of the C_j separates \hat{F} , and we may further cut open along the C_j , obtaining a surface with $m + 1$ components and total genus $g - n$. It follows that there are additional pairwise disjoint simple closed curves E_k , $k = 1, \dots, g - n$, in \hat{F} , reducing \hat{F} to a planar surface of genus 0 when we cut open along the E_k and cap off the resulting $2(g - n)$ boundary curves with disks. Call this latter surface \bar{F} , topologically a 2-sphere with $2n$ holes. Now the C_j separate \bar{F} into $m + 1 = \text{card } S - n + 1$ planar components X_ℓ . As before, each X_ℓ has Euler characteristic ≤ -1 . Therefore, again,

$$\chi(\bar{F}) = 2 - 2n = \sum \chi(X_\ell) \leq (\text{card } S - n + 1)(-1)$$

or equivalently $\text{card } S \leq 3n - 3$, as required.

3. SUFFICIENCY FOR A SINGLE HOMOLOGY CLASS

Here we collect some basic information about the embedding of a single simple closed curve in a surface, and offer an alternative, elementary proof of Theorem 2 for the well-known case of a single homology class.

LEMMA 3.1. *A nonzero homology class $\alpha \in H_1(F)$ is primitive if and only if there exists $\gamma \in H_1(F)$ such that $\gamma \cdot \alpha = 1$.*

Proof. A nonzero element of a finitely generated free abelian group is primitive if and only if it is part of a basis if and only if there is a \mathbf{Z} -valued homomorphism that takes the value 1 on it. Recall that taking intersection numbers of 1-cycles defines a skew symmetric bilinear form on $H_1(F)$. The content of Poincaré Duality in this situation is that this bilinear form is nonsingular, that is, the adjoint homomorphism $H_1(F) \rightarrow \text{Hom}(H_1(F), \mathbf{Z})$ is an isomorphism. The lemma then follows.

LEMMA 3.2. *Any homology class $\alpha \in H_1(F)$ can be represented by an immersed, oriented closed curve on F and also by an embedded, oriented 1-submanifold.*

Proof. The Hurewicz homomorphism $\pi_1(F) \rightarrow H_1(F)$ is onto. Compare W. Massey [1980], Chapter III, Section 7, for example. Any map $S^1 \rightarrow F$ can be approximated by an immersion, with only isolated double points. One can surger any double points, that is, one can replace any pair of small oriented arcs having a single transverse intersection with a pair of parallel oriented arcs with the same end points and lying within a regular neighborhood of the intersecting arcs. In this way one creates a disjoint union of oriented simple closed curves representing the same homology class.

PROPOSITION 3.3. *A homology class α in $H_1(F)$ can be represented by a simple closed curve on F if and only if α is primitive.*

Proof. We sketch a 2-dimensional version of the argument of Bennequin [1977]. If a simple closed curve represents a nonzero homology class, then it is nonseparating. It follows that there is a simple closed curve that meets it transversely in a single point. This implies indivisibility, by the homology invariance of intersection numbers.

For the converse, we may assume that α is nonzero. We begin by representing α by a disjoint union A of oriented simple closed curves, as in Lemma 3.2. We shall assume that A contains the smallest possible number of components and show that this number can be reduced unless it is 1 or it is equal to the divisibility of α .

Cut open F along A —that is, remove the interior of a small tubular neighborhood of A . The boundary of the cut open surface \widehat{F} consists of two copies A_i^+ and A_i^- of each component A_i of A , each of which we orient as the boundary of the orientable surface \widehat{F} . The positive components A_i^+ have the same orientation as A_i , while the negative components A_i^- have the opposite orientation.

If some component R of \widehat{F} contains in its boundary two positive curves A_i^+ and A_j^+ (or two negative curves), then they can be banded together in an orientable way using a band in R . That is, one chooses an embedded arc δ in R meeting A_i^+ and A_j^+ in its two end points only. One then replaces A_i and A_j with the single simple closed curve obtained by removing small arcs in A_i and A_j containing the end points of δ and inserting in their place two embedded arcs parallel to δ . This would reduce the number of components of A . If some component R has boundary just A_i^+ and A_i^- for some A_i ,

then we can conclude that A is connected and we are done. If some R has more than two boundary components, then it contains two positive curves or two negative curves, and we can proceed as above to reduce the number of components of A .

It remains to consider the case where each component R_k of \widehat{F} has exactly two boundary components of the form A_i^+ and A_j^- , where A_i and A_j are distinct components of A . In this case we conclude that we can arrange the components of A in a sequence A_1, A_2, \dots, A_n , so that A_1 is homologous to A_2 , A_2 is homologous to A_3, \dots, A_n is homologous to A_1 . In this case, then, the number n of components is exactly the divisibility of α .

4. SUFFICIENCY FOR INDEPENDENT HOMOLOGY CLASSES

In this section we complete the proof of Theorem 2, dealing with the case of a set of homology classes consisting of independent elements.

LEMMA 4.1. *Let F be a closed orientable surface and let $\alpha_1, \dots, \alpha_n \in H_1(F)$ be independent homology classes that span a summand of $H_1(F)$ on which the intersection pairing of F vanishes. Then there exists $\gamma \in H_1(F)$ such that $\gamma \cdot \alpha_n = 1$ and $\gamma \cdot \alpha_i = 0$ for $i < n$.*

Proof. This is a consequence of Poincaré Duality.

PROPOSITION 4.2. *Let F be a closed orientable surface and let $\alpha_1, \dots, \alpha_n \in H_1(F)$ be independent homology classes that span a summand of $H_1(F)$ on which the intersection pairing of F vanishes. Then there exist pairwise disjoint simple closed curves A_1, \dots, A_n in F representing the homology classes $\alpha_1, \dots, \alpha_n$.*

Proof. The proof will proceed by induction on n . The case $n = 1$ is given by Proposition 3.3.

Now inductively consider the case of $n > 1$ homology classes. By Proposition 3.3 we can find a simple closed curve A_n in F representing α_n . We claim that there is a simple closed curve B_n in F representing a homology class β_n such that B_n meets A_n in exactly one point and such that $[B_n] \cdot \alpha_i = 0$ for $i < n$. By Lemma 4.1 there is a homology class $\gamma_n \in H_1(F)$ such that $\alpha_i \cdot \gamma_n = \delta_{i,n}$. We begin by representing γ_n by a simple closed curve B transverse to A_n . By tubing together neighboring pairs of intersection of B with A_n of opposite sign we can transform B into a disjoint union B'