

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 42 (1996)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: FACTOR EQUIVALENCE RESULTS FOR INTEGERS AND UNITS
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Kapitel: 3. Rings of integers
DOI: <https://doi.org/10.5169/seals-87884>

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(2.5) PROPOSITION. *If N and M are factor equivalent then for any $A[G]$ -linear embedding $j: M \rightarrow N$ the function $H \mapsto [N^H : j(M^H)]_A$ is factorizable.*

Proof. We have $j = \varphi i$, where i is an embedding as in (2.4) and φ is a $K[G]$ -linear automorphism of $N \otimes_A K$. Using [15, Ch. III, § 1, Prop. 2] and the notation of (2.3) we see that

$$[N^H : j(M^H)]_A = d_\varphi(H) \cdot [N^H : i(M^H)]_A .$$

This is a product of two factorizable functions by (2.3) and by our choice of i . \square

The fact that “factor equivalence” is an equivalence relation is an easy consequence of (2.5). If \mathfrak{p} is a prime of K not dividing $\#G$ then condition (1) of (2.4) implies that the \mathfrak{p} -part of $[N^H : i(M^H)]_A$ is factorizable. One can prove this with [16, § 15.2] and [16, § 14.4, Lemma 21].

(2.6) REMARK. The definitions of factorizability given by Fröhlich [8; 9] and Burns [2] for abelian groups G are in agreement with our definitions. They also define the notion called \mathbf{Q} -factorizability in the abelian case, which is a stronger condition than factorizability. However, the function that one wants to be factorizable in the definition of factor equivalence automatically satisfies this stronger condition if it is factorizable. Thus, \mathbf{Q} -factor equivalence is the same as factor equivalence.

In [4, § 3] a factorizable function f with values in $I(\mathbf{Q})$ must also satisfy an additional condition: there should be a map g from the group of complex characters $R_{\mathbf{C}}(G)$ to $I(E)$, where E is some normal number field containing all character values of G , such that g is $\text{Gal}(E/\mathbf{Q})$ -equivariant, and such that $g(1_H^G)$ is the E -ideal generated by $f(H)$. It is not hard to see that this condition is satisfied by all functions that are factorizable in our sense.

3. RINGS OF INTEGERS

Let A be a Dedekind domain with quotient field K of characteristic zero and let L a Galois extension of K with Galois group G . The integral closure B of A in L is again a Dedekind domain. Assume that for all primes of L the residue class field extension is separable.

(3.1) THEOREM. *The $A[G]$ -lattices B and $A[G]$ are factor equivalent.*

Proof. Define a $B[G]$ -module structure on $B \otimes_A B$ by letting B act on the left factor and G on the right. We will show first that $B \otimes_A B$ and $B[G]$ are factor equivalent as $B[G]$ -lattices. Define the canonical $B[G]$ -linear map $\varphi: B \otimes_A B \rightarrow B[G]$ by

$$x \otimes y \mapsto \sum_{\sigma \in G} x\sigma(y) \cdot \sigma^{-1} .$$

Let H be a subgroup of G . If $\sigma_1, \dots, \sigma_n$ are the K -embeddings of L^H in L , and if there is an A -basis $\omega_1, \dots, \omega_n$ of B^H , then the restriction $(B \otimes_A B)^H \rightarrow B[G]^H$ of φ is a B -linear map with matrix $(\sigma_i(\omega_j))_{ij}$ on the bases $\{1 \otimes \omega_j\}$ and $\{b_i\}$, where b_i is the formal sum of those $\sigma \in G$ for which σ^{-1} restricts to σ_i . The square of the determinant of this matrix generates the discriminant $\Delta(B^H/A)$ as an A -ideal. By localization it follows that even if B is not free over A , we have

$$[B[G]^H : \varphi(B \otimes_A B)^H]_B^2 = \Delta(B^H/A) \cdot B .$$

By Hasse's conductor discriminant product formula [15, Ch. VI, §3] the ideal $\Delta(B^H/A)$ is a factorizable function of H , so $B \otimes_A B$ and $B[G]$ are factor equivalent $B[G]$ -lattices.

In order to descend to $A[G]$ -lattices, note that there exists an $A[G]$ -linear injection $i: A[G] \rightarrow B$ by the normal basis theorem, and consider the induced $B[G]$ -linear map $i_*: B[G] \rightarrow B \otimes_A B$ that sends $b\sigma$ to $b \otimes i(\sigma)$ for $b \in B$ and $\sigma \in G$. We have

$$[(B \otimes_A B)^H : i_*(B[G])^H]_B = [B^H : i(A[G])^H]_A \cdot B ,$$

and by (2.5) we know that the left hand side is a factorizable function of H . But then the A -index $[B^H : i(A[G])^H]_A$ is also factorizable. \square

4. S -UNITS

Let L/K be a Galois extension of number fields with Galois group G , and let S be a finite G -stable set of primes of L containing the infinite primes. The ring of S -integers of L consists of all elements of L that are integral outside S . Its class number is written as $h_S(L)$ and its unit group, the group of S -units of L , is denoted by $U_S(L)$. The group of roots of unity in L is denoted by μ_L and its order is written as $w(L)$.