Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 42 (1996)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: CENTRALISERS IN THE BRAID GROUP AND SINGULAR BRAID MONOID

Autor: Fenn, Roger / Zhu, Jun

Kapitel: 4. Centralisers of braid subgroups

DOI: https://doi.org/10.5169/seals-87872

iterated $2r$ times. Noting that I_p persists in w^* it is easy to argue that $w(A * \sigma_j^{2r}) = w(A)$ is impossible; the contradiction. □

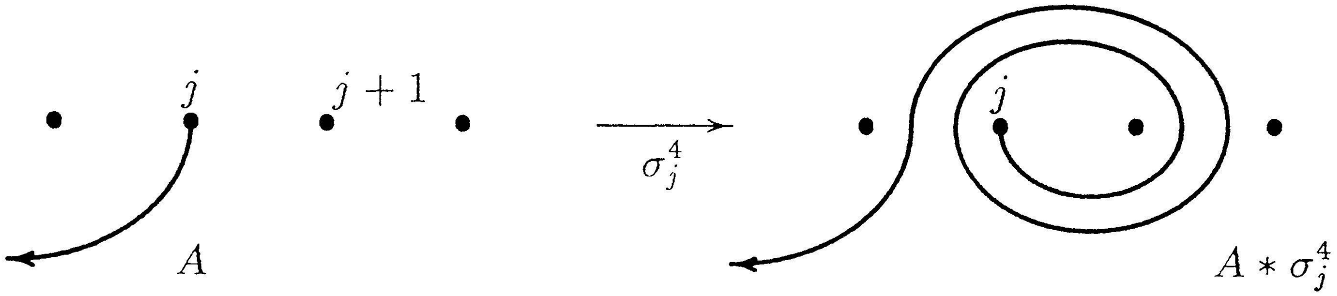

FIGURE 4

The action of $*\sigma_j^{2r}$ on a (j, k)-arc in case $r = 2$

We now turn to the proof of Theorem 2.2. It has already been observed that $(e) \Rightarrow (d) \Rightarrow (a)$, and it is obvious that $(a) \Rightarrow (c) \Rightarrow (b)$. So it remains to establish that $(b) \Rightarrow (e)$. Thus we assume that, for some $r \neq 0$, $\sigma_j^r \beta = \beta \sigma_k^r$. Since the algebraic crossing number of any two strings of a braid is a well-defined braid invariant, this equation is possible only if $\{j, j+1\} * \beta = \{k, k+1\}$. Now, noting that $\beta^{-1} \sigma_j^r \beta = \sigma_k^r$ and that σ_k^r has a (k, k)-band, we conclude that there is a proper ribbon for $\beta^{-1} \sigma_j^r \beta$ from $[k, k+1] \times 0$ to $[k, k+1] \times 1$. Define $A = \beta * [k, k+1] = [k, k+1] * \beta^{-1}$. Then we may assume (possibly after an isotopy) that the planes $\mathbf{C} \times 1/3$ and $\mathbf{C} \times 2/3$ cut the ribbon in the arcs $A \times 1/3$ and $A \times 2/3$. Moreover, the middle third of the ribbon, and Proposition 1.1, imply that $A * \sigma_j^r = A$. By Lemma 3.2, $A = [j, j+1]$ and the theorem is proved. □

4. CENTRALISERS OF BRAID SUBGROUPS

We have established the following.

4.1 THEOREM. *The centraliser in* B_n *of the generator* σ_j *is the subgroup of all braids which have* (j, j)*-bands. This subgroup is isomorphic to* $B_{n-1}^j \times \mathbf{Z}$ *where* B_{n-1}^j *is the subgroup of* B_{n-1} *consisting of all* $(n-1)$*-braids whose permutations stabilise* j. □

The goal of this section is to describe the centraliser of B_r in B_n, $r \leqslant n$, which we will call $C(r, n)$. Here B_r is the r-string braid group with its usual inclusion in B_n, namely as the subgroup generated by $\sigma_1 \ldots \sigma_{r-1}$.

4.2 THEOREM. *The centraliser* $C(r, n)$ *of* B_r *in* B_n *consists of all* n*-braids in which the first* r *strings lie on a ribbon, disjoint from the other strings, and which intersects* $\mathbf{C} \times 0$ *and* $\mathbf{C} \times 1$ *in exactly the straight line intervals from* $[1, r] \times 0$ *and* $[1, r] \times 1$ *(up to isotopy).*

Proof. A braid β is in $C(r, n)$ if and only if it commutes with each σ_j, $1 \leqslant j \leqslant r-1$. Thus $[j, j+1] * \beta = [j, j+1]$, $1 \leqslant j \leqslant r-1$ and so $[1, r] * \beta = [1, r]$, up to isotopy fixing $\{1, \ldots, n\}$. □

It follows that $C(r, n)$ consists of all n-braids constructible as follows. Let $k = n - r + 1$ and consider the subgroup B_k^1 of k-braids whose associated permutation fixes 1. Then replace the first string of a braid in B_k^1 by r parallel strings lying on a ribbon along that string. The ribbon may be twisted by some integral multiple of 2π (or π in the case $r = 2$); such braids are precisely the central elements of B_r.

4.3 THEOREM. *The centraliser* $C(r, n)$ *is isomorphic to the direct product* $B_{n-r+1}^1 \times \mathbf{Z}$. □

A PRESENTATION OF $C(r, n)$. In order to establish a set of generators and defining relations for $C(r, n)$ we recall results of Chow [Ch] regarding B_k^1. This subgroup of B_k is generated by $\sigma_2, \ldots, \sigma_{k-1}$, together with elements $a_2, \ldots, a_k$ defined by

$$a_i := \sigma_1^{-1} \sigma_2^{-1} \cdots \sigma_{i-2}^{-1} \sigma_{i-1}^2 \sigma_{i-2} \cdots \sigma_2 \sigma_1 .$$

These generators satisfy the usual braid relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \qquad |i - j| > 1$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

as well as the following, for $i = 2, \ldots, k-1$:

$$\sigma_i a_j \sigma_i^{-1} = a_j, \qquad j \neq i, i+1$$
$$\sigma_i a_i \sigma_i^{-1} = a_{i+1}$$
$$\sigma_i a_{i+1} \sigma_i^{-1} = a_{i+1}^{-1} a_i a_{i+1} .$$

In fact these are *defining* relations for B_k^1. Chow also noted that the subgroup of B_k^1 generated by the a_i is a normal subgroup (as is clear from the above relations), in fact a *free* group on the generators a_i, and that B_k^1 could be regarded as the semidirect product of that free subgroup with the subgroup generated by $\sigma_2 \ldots \sigma_{k-1}$, the latter group clearly isomorphic with the braid group on $k - 1$ strings.

Applying this to our situation, for each $i = 1, \ldots, n - r$, let A_{r+i} be the n-braid resulting from replacing the first string of the k-braid a_i, defined above, by r parallel strings which lie on an untwisted band. Specifically,

$$A_{r+i} = (\sigma_r^{-1}\sigma_{r+1}^{-1} \cdots \sigma_{r+i-2}^{-1}\sigma_{r+i-1})(\sigma_{r-1}^{-1}\sigma_r^{-1} \cdots \sigma_{r+i-3}^{-1}\sigma_{r+i-2}) \cdots (\sigma_1^{-1}\sigma_2^{-1} \cdots \sigma_{i-1}^{-1}\sigma_i) \times (\sigma_i\sigma_{i-1} \cdots \sigma_1)(\sigma_{i+1}\sigma_i \cdots \sigma_2) \cdots (\sigma_{r+i-1}\sigma_{r+i-2} \cdots \sigma_r)$$

Also let C denote the well-known generator of the centre of the r-string braid group, namely $C = \sigma_1$ if $r = 2$ and in case $r > 2$:

$$C = (\sigma_1\sigma_2 \cdots \sigma_{r-1})^r .$$

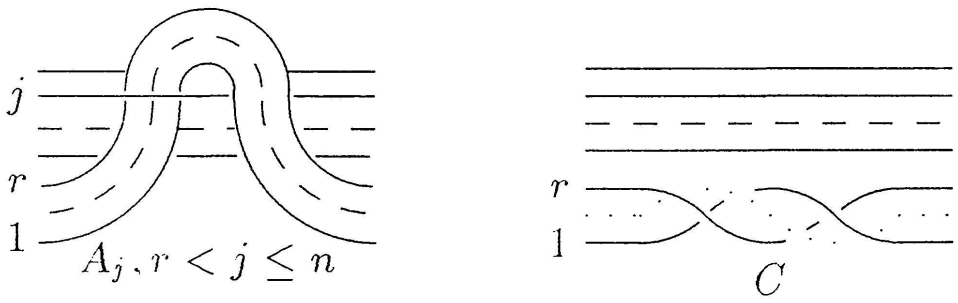

FIGURE 5
Special generators of $C(r, n)$

4.4 THEOREM. *The centraliser $C(r, n)$ of B_r in B_n has the generators:*

$$\sigma_{r+1}, \sigma_{r+2}, \ldots, \sigma_{n-1}, A_{r+1}, \ldots, A_n, C$$

and defining relations:

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad |i - j| > 1$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$
$$\sigma_i A_j \sigma_i^{-1} = A_j, \quad j \neq i, i + 1$$
$$\sigma_i A_i \sigma_i^{-1} = A_{i+1}$$
$$\sigma_i A_{i+1} \sigma_i^{-1} = A_{i+1}^{-1} A_i A_{i+1}$$
$$C\sigma_i = \sigma_i C$$
$$CA_i = A_i C .$$

(Subscripts ranging over all values for which the symbols are in the list of generators.) □