

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 42 (1996)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** CENTRALISERS IN THE BRAID GROUP AND SINGULAR BRAID MONOID  
**Autor:** Fenn, Roger / Zhu, Jun  
**Kapitel:** 4. Centralisers of braid subgroups  
**DOI:** <https://doi.org/10.5169/seals-87872>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

**Download PDF:** 02.04.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

iterated  $2r$  times. Noting that  $I_p$  persists in  $w^*$  it is easy to argue that  $w(A * \sigma_j^{2r}) = w(A)$  is impossible; the contradiction.  $\square$

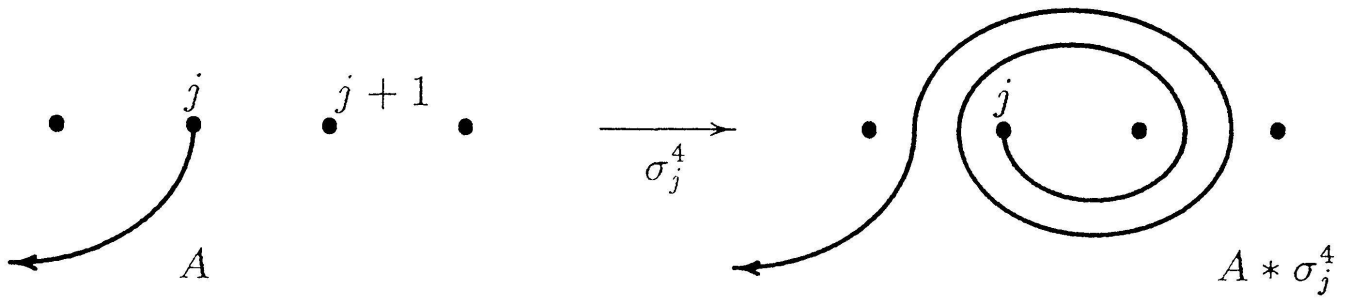


FIGURE 4

The action of  $*\sigma_j^{2r}$  on a  $(j, k)$ -arc in case  $r = 2$

We now turn to the proof of Theorem 2.2. It has already been observed that  $(e) \Rightarrow (d) \Rightarrow (a)$ , and it is obvious that  $(a) \Rightarrow (c) \Rightarrow (b)$ . So it remains to establish that  $(b) \Rightarrow (e)$ . Thus we assume that, for some  $r \neq 0$ ,  $\sigma_j^r \beta = \beta \sigma_k^r$ . Since the algebraic crossing number of any two strings of a braid is a well-defined braid invariant, this equation is possible only if  $\{j, j+1\} * \beta = \{k, k+1\}$ . Now, noting that  $\beta^{-1} \sigma_j^r \beta = \sigma_k^r$  and that  $\sigma_k^r$  has a  $(k, k)$ -band, we conclude that there is a proper ribbon for  $\beta^{-1} \sigma_j^r \beta$  from  $[k, k+1] \times 0$  to  $[k, k+1] \times 1$ . Define  $A = \beta * [k, k+1] = [k, k+1] * \beta^{-1}$ . Then we may assume (possibly after an isotopy) that the planes  $\mathbf{C} \times 1/3$  and  $\mathbf{C} \times 2/3$  cut the ribbon in the arcs  $A \times 1/3$  and  $A \times 2/3$ . Moreover, the middle third of the ribbon, and Proposition 1.1, imply that  $A * \sigma_j^r = A$ . By Lemma 3.2,  $A = [j, j+1]$  and the theorem is proved.  $\square$

#### 4. CENTRALISERS OF BRAID SUBGROUPS

We have established the following.

**4.1 THEOREM.** *The centraliser in  $B_n$  of the generator  $\sigma_j$  is the subgroup of all braids which have  $(j, j)$ -bands. This subgroup is isomorphic to  $B_{n-1}^j \times \mathbf{Z}$  where  $B_{n-1}^j$  is the subgroup of  $B_{n-1}$  consisting of all  $(n-1)$ -braids whose permutations stabilise  $j$ .  $\square$*

The goal of this section is to describe the centraliser of  $B_r$  in  $B_n$ ,  $r \leq n$ , which we will call  $C(r, n)$ . Here  $B_r$  is the  $r$ -string braid group with its usual inclusion in  $B_n$ , namely as the subgroup generated by  $\sigma_1 \dots \sigma_{r-1}$ .

4.2 THEOREM. *The centraliser  $C(r, n)$  of  $B_r$  in  $B_n$  consists of all  $n$ -braids in which the first  $r$  strings lie on a ribbon, disjoint from the other strings, and which intersects  $\mathbf{C} \times 0$  and  $\mathbf{C} \times 1$  in exactly the straight line intervals from  $[1, r] \times 0$  and  $[1, r] \times 1$  (up to isotopy).*

*Proof.* A braid  $\beta$  is in  $C(r, n)$  if and only if it commutes with each  $\sigma_j, 1 \leq j \leq r - 1$ . Thus  $[j, j + 1] * \beta = [j, j + 1], 1 \leq j \leq r - 1$  and so  $[1, r] * \beta = [1, r]$ , up to isotopy fixing  $\{1, \dots, n\}$ .  $\square$

It follows that  $C(r, n)$  consists of all  $n$ -braids constructible as follows. Let  $k = n - r + 1$  and consider the subgroup  $B_k^1$  of  $k$ -braids whose associated permutation fixes 1. Then replace the first string of a braid in  $B_k^1$  by  $r$  parallel strings lying on a ribbon along that string. The ribbon may be twisted by some integral multiple of  $2\pi$  (or  $\pi$  in the case  $r = 2$ ); such braids are precisely the central elements of  $B_r$ .

4.3 THEOREM. *The centraliser  $C(r, n)$  is isomorphic to the direct product  $B_{n-r+1}^1 \times \mathbf{Z}$ .*  $\square$

A PRESENTATION OF  $C(r, n)$ . In order to establish a set of generators and defining relations for  $C(r, n)$  we recall results of Chow [Ch] regarding  $B_k^1$ . This subgroup of  $B_k$  is generated by  $\sigma_2, \dots, \sigma_{k-1}$ , together with elements  $a_2, \dots, a_k$  defined by

$$a_i := \sigma_1^{-1} \sigma_2^{-1} \cdots \sigma_{i-2}^{-1} \sigma_{i-1}^2 \sigma_{i-2} \cdots \sigma_2 \sigma_1 .$$

These generators satisfy the usual braid relations:

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i, & |i - j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \end{aligned}$$

as well as the following, for  $i = 2, \dots, k - 1$ :

$$\begin{aligned} \sigma_i a_j \sigma_i^{-1} &= a_j, & j \neq i, i + 1 \\ \sigma_i a_i \sigma_i^{-1} &= a_{i+1} \\ \sigma_i a_{i+1} \sigma_i^{-1} &= a_{i+1}^{-1} a_i a_{i+1} . \end{aligned}$$

In fact these are *defining* relations for  $B_k^1$ . Chow also noted that the subgroup of  $B_k^1$  generated by the  $a_i$  is a normal subgroup (as is clear from the above relations), in fact a *free* group on the generators  $a_i$ , and that  $B_k^1$  could be regarded as the semidirect product of that free subgroup with the

subgroup generated by  $\sigma_2 \dots \sigma_{k-1}$ , the latter group clearly isomorphic with the braid group on  $k - 1$  strings.

Applying this to our situation, for each  $i = 1, \dots, n - r$ , let  $A_{r+i}$  be the  $n$ -braid resulting from replacing the first string of the  $k$ -braid  $a_i$ , defined above, by  $r$  parallel strings which lie on an untwisted band. Specifically,

$$A_{r+i} = (\sigma_r^{-1} \sigma_{r+1}^{-1} \cdots \sigma_{r+i-2}^{-1} \sigma_{r+i-1}) (\sigma_{r-1}^{-1} \sigma_r^{-1} \cdots \sigma_{r+i-3}^{-1} \sigma_{r+i-2}) \cdots (\sigma_1^{-1} \sigma_2^{-1} \cdots \sigma_{i-1}^{-1} \sigma_i) \times (\sigma_i \sigma_{i-1} \cdots \sigma_1) (\sigma_{i+1} \sigma_i \cdots \sigma_2) \cdots (\sigma_{r+i-1} \sigma_{r+i-2} \cdots \sigma_r)$$

Also let  $C$  denote the well-known generator of the centre of the  $r$ -string braid group, namely  $C = \sigma_1$  if  $r = 2$  and in case  $r > 2$ :

$$C = (\sigma_1 \sigma_2 \cdots \sigma_{r-1})^r .$$

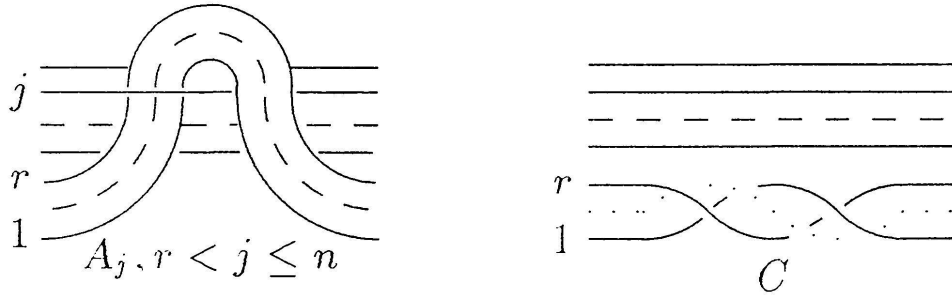


FIGURE 5

Special generators of  $C(r, n)$

4.4 THEOREM. *The centraliser  $C(r, n)$  of  $B_r$  in  $B_n$  has the generators:*

$$\sigma_{r+1}, \sigma_{r+2}, \dots, \sigma_{n-1}, A_{r+1}, \dots, A_n, C$$

and defining relations:

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i, & |i - j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i A_j \sigma_i^{-1} &= A_j, & j \neq i, i + 1 \\ \sigma_i A_i \sigma_i^{-1} &= A_{i+1} \\ \sigma_i A_{i+1} \sigma_i^{-1} &= A_{i+1}^{-1} A_i A_{i+1} \\ C \sigma_i &= \sigma_i C \\ C A_i &= A_i C . \end{aligned}$$

(Subscripts ranging over all values for which the symbols are in the list of generators.)  $\square$