5. THE SINGULAR BRAID MONOID AND THE MAP η

SINGULAR BRAIDS. Just as Vassiliev [Vas] used singular knots to extend and organize knot invariants, it is useful, as in [Bae], [Bir2] to extend the group of braids to the *monoid* of singular braids. (A related construction is given in [FRR].) The strings of a singular braid are allowed to intersect, but only in discrete double points, at which they define a unique common tangent plane. As with braids, one identifies singular braids which are isotopic. The isotopy need not preserve levels, but one must move only through singular braids which have monotone strings, and the tangent plane defined by the two strings at a double point is required to vary smoothly (in 3-space) during any isotopy of the singular braids. Multiplication is by concatenation as with braids; a braid with one or more singularities is not invertible in the monoid. Let SB_n denote the monoid of singular braids on n strings; generators for SB_n are shown in Figure 6.

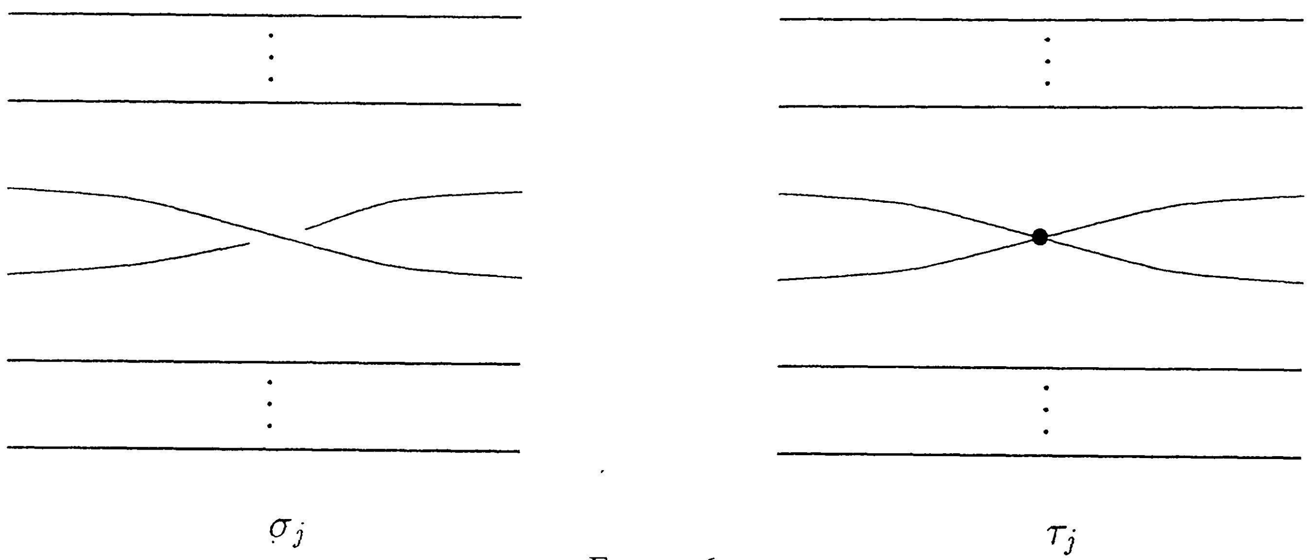

FIGURE 6
Generators of SB_n

In addition to the braid generators $\sigma_1, \cdots, \sigma_{n-1}$ we have the corresponding elementary singular braids $\tau_1, \cdots, \tau_{n-1}$. Together these generate SB_n. A proof is sketched in [Bir2] that, with the invertibility of the σ_i, and the braid relations given in Section 1, the following additional relations serve to define SB_n as a monoid:

$$\sigma_i \tau_i = \tau_i \sigma_i;$$
$$\sigma_i \tau_j = \sigma_j \tau_i, \quad \tau_i \tau_j = \tau_j \tau_i, \qquad |i-j| \geqslant 2;$$
$$\sigma_i \sigma_j \tau_i = \tau_j \sigma_i \sigma_j, \qquad |i-j| = 1 .$$

Notice that the string labels involving a particular singularity are invariant under these relations. Any equivalence between singular braids must match the first singularity involving strings i and j in one braid with the corresponding singularity of the other, etc.

5.1 PROPOSITION. *Left and right cancellation hold in* SB_n, *that is either of* $xy = xz$ *or* $yx = zx$ *with* x, y, z *in* SB_n *implies* $y = z$.

Proof. By symmetry and induction, it is enough to check left cancellation, and the special cases $x = \sigma_j$, which is trivial, or $x = \tau_j$. But if $\tau_j y = \tau_j z$, the singularity of τ_j in each of the two singular braids can be topologically characterised as the one involving the j and $j+1$ strings that is nearest to the left end, so an equivalence taking $\tau_j y$ to $\tau_j z$ must take τ_j to τ_j and therefore y to z. □

Let us take $\mathbf{Z}B_n$ to be the group ring of the braid group B_n. Then the natural map $B_n \to \mathbf{Z}B_n$ can be extended to a monoid homomorphism, $\eta : SB_n \to \mathbf{Z}B_n$ by taking

$$\eta(\sigma_i) = \sigma_i, \ \eta(\tau_i) = \sigma_i - \sigma_i^{-1}.$$

Note that SB_n is a (2-sided) $\mathbf{Z}B_n$-module and η is a $\mathbf{Z}B_n$-homomorphism.

J. Birman in [Bir2] used this homomorphism (with target $\mathbf{C}B_n$) to establish a relation between the Vassiliev knot invariants and quantum group (or generalised Jones) invariants. She conjectured that the kernel of η is trivial, i.e. a nontrivial singular braid in the monoid SB_n never maps to zero in $\mathbf{Z}B_n$. A stronger conjecture is that η is an embedding. The weak version of Birman's conjecture (as actually stated) is rather easy — we give the proof below. The injectivity question seems much harder, and is still an open question at the time of this writing. In the next section we will apply techniques developed in the previous sections to show that η is injective at least when restricted to singular braids with no more than two singularities. We understand that Birman also has obtained these results independently.

To analyse the η map, it is useful to consider the *degree* of a (singular) braid, by which we mean the total exponent sum of all the σ_j in an expression of the braid in terms of generators. It is well-known, and easy to see from the homogeneity of the braid relations, that degree is well-defined. Likewise, the number of singularities is invariant and we define $SB_n^{(t)} \subset SB_n$ as the subset of singular braids with exactly t singularities. The following is routine to verify.

5.2 PROPOSITION. *Suppose* $x \in SB_n^{(t)}$ *is a singular braid of degree* s. *Then* $\eta(x) \in \mathbf{Z}B_n$ *is a linear combination of* 2^t *elements of* B_n *(call them terms). There is a unique term of maximal degree* $s + t$ *and a unique term of minimal degree* $s - t$. *More generally, for each integer* $u, 0 \leqslant u \leqslant t, \eta(x)$ *has* $\binom{t}{u}$ *terms of degree* $s + t - 2u$, *and each of these terms has coefficient* $(-1)^u$. □

There may be some cancellation among the terms of degree strictly between $s - t$ and $s + t$, but since there is only one term of maximum and one term of minimal degree, they cannot be cancelled and we draw the following conclusions.

5.3 COROLLARY. *No element of* SB_n *maps to zero under* η. □

The kernel of η is also trivial in another sense.

5.4 COROLLARY. *If* $1 \in B_n \subset SB_n$ *denotes the identity braid, then* $\eta^{-1}(1) = 1$. □

To close this section we consider the natural extension of η to the monoid ring $\mathbf{Z}SB_n$.

5.5 PROPOSITION. *The extension* $\eta : \mathbf{Z}SB_n \to \mathbf{Z}B_n$ *is not injective.*

Proof. τ_1 and $\sigma_1 - \sigma_1^{-1}$ are two elements of $\mathbf{Z}SB_n$ with the same image. For a more subtle example, consider the elements

$$x = \tau_1\tau_2\sigma_1^{-1} + \tau_1\sigma_2\tau_1 , \quad y = \tau_2\sigma_1^{-1}\tau_2 + \sigma_2\tau_1\tau_2 .$$

An easy calculation verifies that $\eta(x) = \eta(y)$. However, $x \neq y$, as can be seen by examining their images under the map $\tau_i \to \sigma_i, \sigma_i \to \sigma_i$. □

The above example is related to certain canonical relations obeyed by the Vassiliev invariants — see [Bir2], p. 274, or [Bar].

6. RESULTS REGARDING INJECTIVITY OF η

Note that if $x, y \in SB_n$ satisfy $\eta(x) = \eta(y)$, then they both have the same number of singularities, i.e. $x \in SB_n^{(t)}$ if and only if $y \in SB_n^{(t)}$. The relevance of bands to the injectivity question will be illustrated by first checking