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Artikel: INTRODUCTORY NOTES ON RICHARD THOMPSON'S GROUPS
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$$\theta(p^{-1}q) = \theta(C_n^m)$$
$$\text{and} \quad \theta\big((p^{-1}q)^{n+2}\big) = \theta\big((C_n^m)^{n+2}\big) = \theta\big((C_n^{n+2})^m\big) \overset{5.6.v)}{=} \theta(1) = 1\,.$$

By Lemma 5.4, there is a homomorphism $\alpha\colon F \to T_1/N$ defined on generators by $\alpha(A) = \theta(A)$ and $\alpha(B) = \theta(B)$. If $p^{-1}q \neq 1$, then $(p^{-1}q)^{n+2} \neq 1$, and so $\alpha(F)$ is a proper quotient group of F. Since every proper quotient group of F is Abelian by Theorem 4.3, $\theta(AB) = \theta(BA)$. If $p^{-1}q = 1$, then $m, n > 0$ and $1 = \theta(C_n^m) \overset{5.6.iv)}{=} \theta(C_{n+1}^{m+1})\,\theta(X_{m-1}^{-1})$ and hence $\theta(X_{m-1}^{n+3}) = \theta\big((C_{n+1}^{m+1})^{n+3}\big) = \theta\big((C_{n+1}^{n+3})^{m+1}\big) = \theta(1) = 1$. It follows as before that $\theta(AB) = \theta(BA)$. Hence $\theta(A^{-1}BA) = \theta(B)$, so $\theta(A^{-1}C) = \theta(BA^{-2}C)$ by relation 4). Hence $\theta(BA^{-1}) = 1$, and so $\theta(B) = 1$ by relation 3). This implies that $\theta(A) = 1$. It now follows from relation 5) that $\theta(C) = 1$. Thus $N = T_1$, and so T_1 is simple. □

COROLLARY 5.9. *T_1 is isomorphic to T.*

§6. THOMPSON'S GROUP V

As with the previous section, the material in this section is mainly from unpublished notes of Thompson [T1]; [T1] contains the statements of the lemmas (except for Lemma 6.2) and the statement and proof of Theorem 6.9, but does not contain the proofs of the lemmas.

Let V be the group of right-continuous bijections of S^1 that map images of dyadic rational numbers to images of dyadic rational numbers, that are differentiable except at finitely many images of dyadic rational numbers, and such that, on each maximal interval on which the function is differentiable, the function is linear with derivative a power of 2. As before, it is easy to prove that V is a group.

We can associate tree diagrams with elements of V as we did for F and T, except that now we need to label the leaves of the domain and range trees to indicate the correspondence between the leaves. For example, reduced tree diagrams for A, B, and C are given in Figure 16.

Using the identification of S^1 as the quotient of $[0,1]$, define $\pi_0 : S^1 \to S^1$ by

$$\pi_0(x) = \begin{cases} \frac{x}{2} + \frac{1}{2}, & 0 \le x < \frac{1}{2} \\ 2x - 1, & \frac{1}{2} \le x < \frac{3}{4} \\ x, & \frac{3}{4} \le x < 1. \end{cases}$$

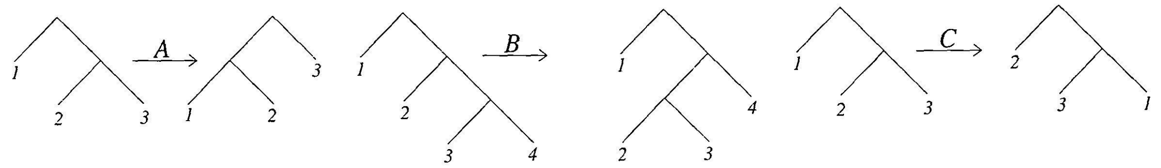

FIGURE 16
Reduced tree diagrams for A, B, and C

We define elements X_n and C_n of V as before. That is, $X_0 = A$, $X_n = A^{-n+1}BA^{n-1}$ for an integer $n \geq 1$, and $C_n = A^{-n+1}CB^{n-1}$ for an integer $n \geq 1$. Define π_n, $n \geq 1$, by $\pi_1 = C_2^{-1}\pi_0 C_2$ and $\pi_n = A^{-n+1}\pi_1 A^{n-1}$ for $n \geq 2$. Reduced tree diagrams from π_0, π_1, π_2, and π_3 are given in Figure 17.

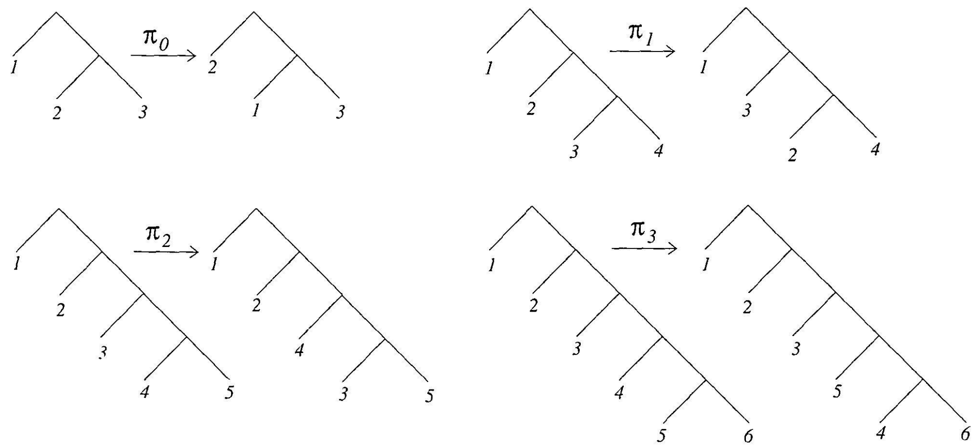

FIGURE 17
Reduced tree diagrams for π_i, $0 \leq i \leq 3$

It is easy to see for every positive integer n that $\pi_0, \ldots, \pi_{n-1}$ generate a subgroup of V isomorphic with the symmetric group of all permutations of the $n+1$ intervals $[0, 1-2^{-1}]$, $[1-2^{-1}, 1-2^{-2}]$, $[1-2^{-2}, 1-2^{-3}], \ldots,$ $[1-2^{-n}, 1-2^{-(n+1)}]$. Furthermore $\pi_0, \ldots, \pi_{n-1}$ and C_n generate a subgroup of V isomorphic with the symmetric group of all permutations of the $n+2$ intervals $[0, 1-2^{-1}]$, $[1-2^{-1}, 1-2^{-2}]$, $[1-2^{-2}, 1-2^{-3}], \ldots,$ $[1-2^{-n}, 1-2^{-(n+1)}]$, $[1-2^{-(n+1)}, 1]$ for every positive integer n.

LEMMA 6.1. *The elements A, B, C, and π_0 generate V and satisfy the following relations:*

1) $[AB^{-1}, X_2] = 1$;
2) $[AB^{-1}, X_3] = 1$;
3) $C_1 = BC_2$;
4) $C_2X_2 = BC_3$;

5) $C_1 A = C_2^2$;

6) $C_1^3 = 1$;

7) $\pi_1^2 = 1$;

8) $\pi_1 \pi_3 = \pi_3 \pi_1$;

9) $(\pi_2 \pi_1)^3 = 1$;

10) $X_3 \pi_1 = \pi_1 X_3$;

11) $\pi_1 X_2 = B \pi_2 \pi_1$;

12) $\pi_2 B = B \pi_3$;

13) $\pi_1 C_3 = C_3 \pi_2$; *and*

14) $(\pi_1 C_2)^3 = 1$.

Proof. Let H be the subgroup of V generated by A, B, C, and π_0. To prove that $H = V$, it suffices to prove that if R and S are $\mathcal{T}$-trees with n leaves labeled by $1, \ldots, n$, then there is an element of H with domain tree R and range tree S which preserves labels. Since H is a group and A and B generate the subgroup F of V, we can assume that $R = S = \mathcal{T}_{n-1}$. So assume that $R = S = \mathcal{T}_{n-1}$. Each element of the subgroup of V generated by π_0 and C_{n-2} has a tree diagram with domain tree and range tree $\mathcal{T}_{n-1}$, and this subgroup is isomorphic to the symmetric group Σ_n, acting on the leaves of $\mathcal{T}_{n-1}$. Hence there is an element of V with domain tree R and range tree S which preserves labels, and $H = V$.

It follows from Lemma 5.2 that relations 1)-6) are satisfied. Relations 7), 8), 9), 13), and 14) follow easily from the viewpoint of permutations. Relation 10) is true because the supports of π_1 and X_3 are disjoint. Relations 11) and 12) can be established by verifying that the reduced tree diagrams for the two elements are the same; the tree diagrams are computed in Figures 18 and 19. □

The group V_1 will be defined via generators and relators. There will be four generators, A, B, C, and π_0. We introduce words X_n, C_n, and π_n as before. That is, $X_0 = A$, $X_n = A^{-n+1} B A^{n-1}$ for an integer $n \geq 1$, $C_n = A^{-n+1} C B^{n-1}$ for an integer $n \geq 1$, $\pi_1 = C_2^{-1} \pi_0 C_2$, and $\pi_n = A^{-n+1} \pi_1 A^{n-1}$ for $n \geq 2$.

Let

$$V_1 = \langle A, B, C, \pi_0 : [AB^{-1}, X_2], [AB^{-1}, X_3], BC_2(C_1)^{-1}, BC_3(C_2X_2)^{-1}, C_2^2(C_1A)^{-1}, C_1^3, \pi_1^2, \pi_3\pi_1(\pi_1\pi_3)^{-1}, (\pi_2\pi_1)^3, \pi_1X_3(X_3\pi_1)^{-1}, B\pi_2\pi_1(\pi_1X_2)^{-1}, B\pi_3(\pi_2B)^{-1}, C_3\pi_2(\pi_1C_3)^{-1}, (\pi_1C_2)^3\rangle$$

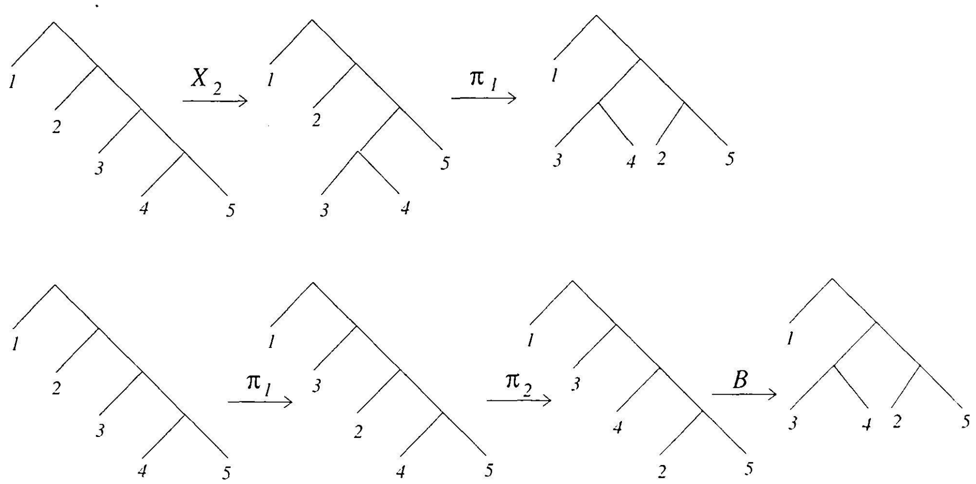

FIGURE 18

Reduced tree diagrams for $\pi_1 X_2$ and $B\pi_2\pi_1$

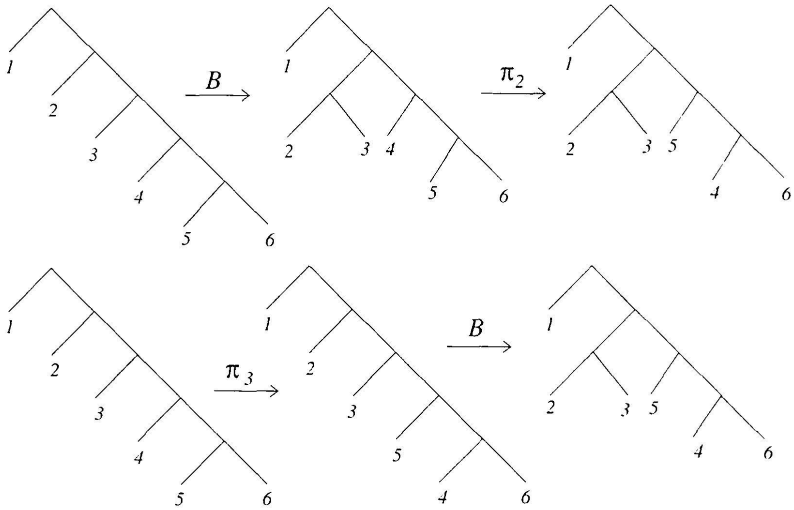

FIGURE 19

Reduced tree diagrams for $\pi_2 B$ and $B\pi_3$

We will prove that V_1 is simple. Since there is a surjection from V_1 to V by Lemma 6.1, it will follow that $V_1 \cong V$ and V is simple.

Lemmas 6.3-6.8 contain the relations we need among the π_i's, the X_i's, and the C_i's. Lemma 6.2 isolates some parts of them that will be needed in the proof of Lemma 6.3.

LEMMA 6.2. *Let i be a positive integer and let j be a integer.*

i) If $0 \le j < i$, then $\pi_i X_j = X_j \pi_{i+1}$.

ii) If $j \ge i+2$, then $\pi_i X_j = X_j \pi_i$.

iii) If $i > j > 0$, then $C_i \pi_j = \pi_{j-1} C_i$.

Proof. We begin the proof of i) by proving that AB^{-1} commutes with X_n and π_n for every integer $n \geq 2$. For this let H be the centralizer of AB^{-1} in V_1. Theorem 3.4 easily implies that H contains X_n for every integer $n \geq 2$. We prove that $\pi_n \in H$ for every integer $n \geq 2$ by induction on n. For $n = 2$ we have $\pi_3 = A^{-1}\pi_2 A$, and the relator $B\pi_3(\pi_2 B)^{-1}$ gives $\pi_3 = B^{-1}\pi_2 B$. Hence $\pi_2 \in H$. Now let n be an integer with $n \geq 2$, and suppose that $\pi_n \in H$. Since H contains π_n, X_n, and X_{n+1}, $A^{n-1}HA^{-n+1}$ contains π_1, X_1, and X_2. Thus the relator $B\pi_2\pi_1(\pi_1 X_2)^{-1}$ easily gives $\pi_2 \in A^{n-1}HA^{-n+1}$, and so $\pi_{n+1} \in H$. This proves that AB^{-1} commutes with X_n and π_n for every integer $n \geq 2$.

We now prove i) by induction on j. If $j = 0$, then i) is clear. Suppose that $j = 1$ and that i is an integer with $i > 1$. We have $A^{-1}\pi_i A = \pi_{i+1}$, and the previous paragraph shows that $AB^{-1}\pi_i BA^{-1} = \pi_i$. These identities imply that $B^{-1}\pi_i B = \pi_{i+1}$, which gives ii) when $j = 1$. Now suppose that $j > 1$ and that i is an integer with $i > j$. We have $\pi_{i-j+1}X_1 = X_1\pi_{i-j+2}$, and so $A^{-j+1}\pi_{i-j+1}A^{j-1}A^{-j+1}X_1A^{j-1} = A^{-j+1}X_1A^{j-1}A^{-j+1}\pi_{i-j+2}A^{j-1}$. Hence $\pi_i X_j = X_j\pi_{i+1}$. This proves i).

Since $\pi_1 X_3 = X_3\pi_1$, $\pi_2 X_4 = A^{-1}\pi_1 X_3 A = A^{-1}X_3\pi_1 A = X_4\pi_2$. $B\pi_2\pi_1 X_4 = \pi_1 X_2 X_4 = \pi_1 X_3 X_2 = X_3\pi_1 X_2 = X_3 B\pi_2\pi_1 = BX_4\pi_2\pi_1 = B\pi_2 X_4\pi_1$, and so $\pi_1 X_4 = X_4\pi_1$. If $n \geq 4$ and $\pi_1 X_n = X_n\pi_1$, then $X_3\pi_1 X_{n+1} = \pi_1 X_3 X_{n+1} = \pi_1 X_n X_3 = X_n\pi_1 X_3 = X_n X_3\pi_1 = X_3 X_{n+1}\pi_1$ and so $\pi_1 X_{n+1} = X_{n+1}\pi_1$. Hence it follows by induction that $\pi_1 X_j = X_j\pi_1$ if $j \geq 3$. If i, j are positive integers and $j \geq i + 2$, $\pi_i X_j = A^{-i+1}\pi_1 A^{i-1}A^{-i+1}X_{j-i+1}A^{i-1} = A^{-i+1}\pi_1 X_{j-i+1}A^{i-1} = A^{-i+1}X_{j-i+1}\pi_1 A^{i-1} = X_j\pi_i$. This proves ii).

We prove iii) by induction on j and i. We have $C_3\pi_2 = \pi_1 C_3$. If $2 < i$ and $C_i\pi_2 = \pi_1 C_i$, then $X_i C_{i+1}\pi_2 = C_i\pi_2 = \pi_1 C_i = \pi_1 X_i C_{i+1} = X_i\pi_1 C_{i+1}$ and hence $C_{i+1}\pi_2 = \pi_1 C_{i+1}$. It follows by induction on i that $C_i\pi_2 = \pi_1 C_i$ if $i > 2$. If $1 < j < i$ and $C_i\pi_j = \pi_{j-1}C_i$, then $C_{i+1}\pi_{j+1} = C_{i+1}B^{-1}B\pi_{j+1} = C_{i+1}B^{-1}\pi_j B = A^{-1}C_i BB^{-1}\pi_j B = A^{-1}C_i\pi_j B = A^{-1}\pi_{j-1}C_i B = A^{-1}\pi_{j-1}AA^{-1}C_i B = \pi_j C_{i+1}$. It follows by induction on j that $C_i\pi_j = \pi_{j-1}C_j$ if $1 < j < i$.

To finish the proof of iii), it remains to show that $C_i\pi_1 = \pi_0 C_i$ if $1 < i$. Since $\pi_1 = C_2^{-1}\pi_0 C_2$, $C_2\pi_1 = \pi_0 C_2$. Suppose $i \geq 2$ and $C_i\pi_1 = \pi_0 C_i$. Since $C_i A = C_{i+1}^2$ and $\pi_1 A = A\pi_2$, $C_{i+1}^2\pi_2 = C_i A\pi_2 = C_i\pi_1 A = \pi_0 C_i A = \pi_0 C_{i+1}^2$. But $\pi_1 C_{i+1} = C_{i+1}\pi_2$, so $C_{i+1}\pi_1 C_{i+1} = C_{i+1}^2\pi_2 = \pi_0 C_{i+1}^2$ and hence $C_{i+1}\pi_1 = \pi_0 C_{i+1}$. It follows by induction that $C_i\pi_1 = \pi_0 C_i$ if $1 < i$. □

LEMMA 6.3. *If i is a nonnegative integer, then*

i) $\pi_i^2 = 1$,

ii) $(\pi_{i+1}\pi_i)^3 = 1$, *and*

iii) $\pi_i\pi_j = \pi_j\pi_i$ *if* $j \geq i+2$.

Proof. $\pi_1^2 = 1$ from the definition of V_1, and since the π_i's are conjugate to each other, $\pi_i^2 = 1$ for $i \geq 0$.

$(\pi_2\pi_1)^3 = 1$ is one of the defining relations. Lemma 6.2.iii) shows that $\pi_{i+1}\pi_i$ is conjugate to $\pi_2\pi_1$ for every nonnegative integer i. Hence $(\pi_{i+1}\pi_i)^3 = 1$ for every nonnegative integer i. This proves ii).

We may likewise use Lemma 6.2.iii) to reduce the proof of iii) to the case in which $i = 1$. Since $\pi_1\pi_3 = \pi_3\pi_1$, $\pi_2\pi_4 = A^{-1}\pi_1\pi_3A = A^{-1}\pi_3\pi_1A = \pi_4\pi_2$. Since $\pi_1\pi_3 = \pi_3\pi_1$, $\pi_1\pi_3X_2 = \pi_3\pi_1X_2$, $\pi_1X_2\pi_4 = \pi_3X_1\pi_2\pi_1$, $X_1\pi_2\pi_1\pi_4 = X_1\pi_4\pi_2\pi_1 = X_1\pi_2\pi_4\pi_1$, and hence $\pi_1\pi_4 = \pi_4\pi_1$. If $n \geq 4$ and $\pi_1\pi_n = \pi_n\pi_1$, then $X_3\pi_1\pi_{n+1} = \pi_1X_3\pi_{n+1} = \pi_1\pi_nX_3 = \pi_n\pi_1X_3 = \pi_nX_3\pi_1 = X_3\pi_{n+1}\pi_1$. It follows by induction that $\pi_1\pi_j = \pi_j\pi_1$ if $j \geq 3$. This proves iii).

LEMMA 6.4. *If i and j are nonnegative integers, then*

i) $\pi_iX_j = X_j\pi_i$ *if* $j \geq i+2$,

ii) $\pi_iX_{i+1} = X_i\pi_{i+1}\pi_i$,

iii) $\pi_iX_i = X_{i+1}\pi_i\pi_{i+1}$, *and*

iv) $\pi_iX_j = X_j\pi_{i+1}$ *if* $0 \leq j < i$.

Proof. If $i > 0$, then i) is Lemma 6.2.ii). For $i = 0$ suppose that n is an integer with $j < n$. Then $\pi_0X_jC_{n+1} = \pi_0C_nX_{j+1} = C_n\pi_1X_{j+1} = C_nX_{j+1}\pi_1 = X_jC_{n+1}\pi_1 = X_j\pi_0C_{n+1}$ by Lemmas 5.5.ii), 6.2.iii), and 6.2.ii). Hence $\pi_0X_j = X_j\pi_0$ if $j \geq 2$. This proves i).

For ii), the case $i = 1$ is one of the defining relations. Since $\pi_1X_2 = B\pi_2\pi_1$, Lemmas 5.5.ii) and 6.2.iii) give that $\pi_0BC_3 = \pi_0C_2X_2 = C_2\pi_1X_2 = C_2B\pi_2\pi_1 = AC_3\pi_2\pi_1 = A\pi_1\pi_0C_3$. This implies that $\pi_0B = A\pi_1\pi_0$, which gives ii) when $i = 0$. If $i > 1$, then conjugating the relation $\pi_1X_2 = X_1\pi_2\pi_1$ by A^{i-1} gives $\pi_iX_{i+1} = X_i\pi_{i+1}\pi_i$. This proves ii).

iii) follows immediately from ii) since each π_i has order 2.

iv) is Lemma 6.2.i). □

LEMMA 6.5. *Let n and k be positive integers with $n > k$. Then*

i) $C_n\pi_k = \pi_{k-1}C_n$,

ii) $C_n\pi_0 = \pi_0\cdots\pi_{n-1}C_n^2$,

iii) $C_n^2\pi_0 = \pi_{n-1}\cdots\pi_0C_n$, *and*

iv) $C_n^3\pi_0 = \pi_{n-1}C_n^3$.

Proof. i) is Lemma 6.2.iii).

We prove ii) by induction. Since $(\pi_1 C_2)^3 = 1$, $(C_2\pi_1)^3 = 1$. This implies that $C_2\pi_1 C_2 = \pi_1 C_2^{-1}\pi_1$, and hence that $C_2^2\pi_1 C_2^{-1} = C_2\pi_1 C_2^{-1}\pi_1 C_2^2$ by Lemma 5.6.v). Hence $C_2\pi_0 = C_2(C_2\pi_1 C_2^{-1}) = (C_2\pi_1 C_2^{-1})\pi_1 C_2^2 = \pi_0\pi_1 C_2^2$, which proves ii) when $n = 2$. Suppose that $n \geq 2$ and $C_n\pi_0 = \pi_0 \cdots \pi_{n-1} C_n^2$. Then

$$\begin{aligned} X_n C_{n+1}\pi_0 &= C_n\pi_0 = \pi_0 \cdots \pi_{n-1} C_n^2 = \pi_0 \cdots \pi_{n-1} C_n X_n C_{n+1} \\ &= \pi_0 \cdots \pi_{n-1} X_{n-1} C_{n+1}^2 = \pi_0 \cdots \pi_{n-2} X_n \pi_{n-1}\pi_n C_{n+1}^2 \\ &= X_n \pi_0 \cdots \pi_{n-2}\pi_{n-1}\pi_n C_{n+1}^2 , \end{aligned}$$

and hence $C_{n+1}\pi_0 = \pi_0 \cdots \pi_n C_{n+1}^2$. ii) now follows by induction.

iii) follows from ii):

$$C_n = (C_n\pi_0)\pi_0 = (\pi_0 \cdots \pi_{n-1} C_n^2)\pi_0 ,$$

so $C_n^2\pi_0 = (\pi_0 \cdots \pi_{n-1})^{-1} C_n = \pi_{n-1} \cdots \pi_0 C_n$.

iv) follows from i), ii), and iii):

$$\begin{aligned} C_n^3\pi_0 &= C_n(C_n^2\pi_0) = C_n(\pi_{n-1} \cdots \pi_0 C_n) = \pi_{n-2} \cdots \pi_0 C_n(\pi_0 C_n) \\ &= \pi_{n-2} \cdots \pi_0(\pi_0 \cdots \pi_{n-1} C_n^2) C_n = \pi_{n-1} C_n^3 . \qquad \square \end{aligned}$$

LEMMA 6.6. *Let k, m, and n be integers with $0 \leq m < n+2$ and $0 \leq k < n$. Then*

i) if $m \leq k$, $C_n^m \pi_k = \pi_{k-m} C_n^m$,

ii) if $m = k+1$, $C_n^m \pi_k = \pi_0 \cdots \pi_{n-1} C_n^{m+1}$,

iii) if $m = k+2$, $C_n^m \pi_k = \pi_{n-1} \cdots \pi_0 C_n^{m-1}$, and

iv) if $m > k+2$, $C_n^m \pi_k = \pi_{k+(n+2-m)} C_n^m$.

Proof. i) follows from Lemma 6.5.i) by induction.

Now consider ii). If $n \geq 2$ and $m = k+1$, then by Lemmas 6.6.i) and 6.5.ii) $C_n^m\pi_k = C_n C_n^k \pi_k = C_n\pi_0 C_n^k = \pi_0 \cdots \pi_{n-1} C_n^2 C_n^k = \pi_0 \cdots \pi_{n-1} C_n^{m+1}$, which proves ii) if $n \geq 2$. By Lemmas 5.6.i), 6.3.i), and 6.5.iv), $C^2 B = C_2^3 = \pi_1^2 C_2^3 = \pi_1 C_2^3 \pi_0$. Hence $C^2 B\pi_0 = \pi_1 C_2^3 = \pi_0\pi_0\pi_1 C_2^3 = \pi_0 C_2\pi_0 C_2 = \pi_0 C_2^2 \pi_1$ by Lemmas 6.3.i), 6.5.ii), and 6.5.i). Hence $C^2 B\pi_0\pi_1 = \pi_0 C_2^2$, and so $C^2\pi_0 A = \pi_0 CA$ by Lemmas 6.4.iii) and 5.5.iii). This gives $C^2\pi_0 = \pi_0 C$, and hence $C\pi_0 = C(\pi_0 C)C^{-1} = C(C^2\pi_0)C^{-1} = \pi_0 C^2$. This completes the proof of ii).

If $n = 1$, then the assumptions of iii) imply that $k = 0$ and $m = 2$, and so iii) becomes $C_1^2\pi_0 = \pi_0 C_1$, hence $C^2\pi_0 = \pi_0 C$. This was proved in the above paragraph. If $n \geq 2$ and $m = k+2$, then by Lemmas 6.6.i) and 6.5.iii) $C_n^m\pi_k = C_n^2 C_n^k \pi_k = C_n^2\pi_0 C_n^k = \pi_{n-1}\cdots\pi_0 C_n C_n^k = \pi_{n-1}\cdots\pi_0 C_n^{m-1}$, which proves iii).

To prove iv), suppose that $m > k+2$. Then by Lemmas 6.6.i) and 6.5.iv) $C_n^m\pi_k = C_n^{m-k-3}C_n^3 C_n^k\pi_k = C_n^{m-k-3}C_n^3\pi_0 C_n^k = C_n^{m-k-3}\pi_{n-1}C_n^{k+3} = \pi_{n-1-(m-k-3)}C_n^m$, which proves iv). □

For each positive integer n, let $\Pi(n)$ be the subgroup of V_1 generated by $\{\pi_0, \dots, \pi_{n-1}\}$, and let $\Pi = \cup_{n\in\mathbf{N}}\Pi(n)$.

Let Σ be the group of permutations of $\mathbf{N}$ with finite support. Then

$$\Sigma = \langle s_0, s_1, s_2, \dots : (s_i)^2 \text{ for all } i,\ (s_i s_{i+1})^3 \text{ for all } i,\ (s_i s_j)^2 \text{ for all } i \text{ and all } j \geq i+2 \rangle .$$

Furthermore, in every proper quotient group of Σ, the image of s_0 is the image of s_1. Since Π is a quotient group of Σ and $\pi_0 \neq \pi_1$ in V, Π is isomorphic to Σ.

Following the terminology for F, an element of V_1 which is a product of nonnegative powers of the X_i's will be called *positive* and an inverse of a positive element will be called *negative*.

LEMMA 6.7. *If p is a positive element of V_1 and $\pi \in \Pi$, then $\pi p = p'\pi'$ for some positive element p' and some $\pi' \in \Pi$.*

Proof. Lemma 6.7 follows from Lemma 6.4. □

LEMMA 6.8.

i) If m, n are positive integers with $m < n+2$ and if $\pi \in \Pi(n)$, then $C_n^m\pi = \pi' C_n^{m'}$ for some $\pi' \in \Pi(n)$ and some positive integer m' with $m' < n+2$.

ii) For each $n \in \mathbf{N}$, the subgroup of V_1 generated by $\Pi(n)$ and C_n is finite.

Proof. i) follows from Lemmas 6.6 and 5.6.v). ii) follows from i) and Lemma 5.6.v). □

THEOREM 6.9. *V_1 is simple.*

Proof. Suppose N is a nontrivial normal subgroup of V_1, and let $\theta\colon V_1 \to V_1/N$ be the quotient homomorphism. Then there is an element $g \in V_1$ with $g \neq 1$ and $\theta(g) = 1$. By Lemmas 5.6.iii), 5.6.iv), 6.7, 6.8.i) and Theorem 5.7 we have $g = p\pi C_n^m q^{-1}$ for some positive elements p and q, some integers m, n with $0 \leq m < n+2$, and some element $\pi \in \Pi(n)$. Then $\theta(\pi C_n^m) = \theta(p^{-1}q)$. Lemma 6.8.ii) implies that πC_n^m has finite order, say, k. Furthermore the subgroup of V_1 generated by A and B is torsion-free because it maps injectively to $F \subseteq V$ by Theorem 3.4. Hence either $(p^{-1}q)^k \neq 1$ and $\theta\big((p^{-1}q)^k\big) = 1$ or $\pi C_n^m \neq 1$ and $\theta(\pi C_n^m) = 1$. Suppose that $\pi C_n^m \neq 1$ and $\theta(\pi C_n^m) = 1$. If $m = 0$, then $\pi \neq 1$ and $\theta(\pi) = 1$. This implies that $\theta(\pi_0) = \theta(\pi_1)$, and hence by Lemma 6.5 that $\theta(\pi_0 C_2) = \theta(C_2\pi_1) = \theta(C_2\pi_0) = \theta(\pi_0\pi_1 C_2^2)$. But then $\theta(\pi_1 C_2) = 1$, so we may assume that $m > 0$. Next suppose that $m > 0$. Then $\pi C_n^m = \pi X_{n+1-m} C_{n+1}^m$ by Lemma 5.6.iii). Lemma 6.4 implies that there exists a nonnegative integer i and $\pi' \in \Pi(n+1)$ such that $\pi C_n^m = X_i \pi' C_{n+1}^m$. Thus we are in the above case in which $(p^{-1}q)^k \neq 1$ and $\theta\big((p^{-1}q)^k\big) = 1$.

In each case there is an element $h \in V_1$ such that $h \neq 1$, $\theta(h) = 1$, and h can be represented as a word in $A^{\pm 1}$, $B^{\pm 1}$, and $C^{\pm 1}$. Let $\alpha\colon T_1 \to V_1/N$ be the homomorphism defined by $\alpha(A) = \theta(A)$, $\alpha(B) = \theta(B)$, and $\alpha(C) = \theta(C)$. Then there is an element $h' \in T_1$ with $h' \neq 1$ and $\alpha(h') = 1$. Since T_1 is simple by Theorem 5.8, $\theta(A) = \theta(B) = \theta(C) = 1$. Because π_i and π_j are conjugate via a power of A, $\theta(\pi_i) = \theta(\pi_j)$ for all nonnegative integers i and j. By Lemma 6.6.ii) with $k = 1$, $m = 2$ and $n = 2$, $\theta(\pi_1) = \theta(C_2^2\pi_1) = \theta(\pi_0\pi_1 C_2^3) = \theta(\pi_0\pi_1)$, and hence $\theta(\pi_0) = 1$. This implies that the quotient group is trivial. □

§7. PIECEWISE INTEGRAL PROJECTIVE STRUCTURES

The definition of piecewise integral projective structures is due to W. Thurston. These structures arise naturally on the boundaries of Teichmüller spaces of surfaces. The interpretations of F and T as groups of piecewise integral projective homeomorphisms are also due to Thurston; we learned this from him in 1975. Greenberg [Gr] used this interpretation in his study of these groups.

Fix a positive integer n.