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$$I_\alpha := \prod_{i=1}^{n} [|\alpha_{2i} - \alpha_{2i-1}|, \alpha_{2i} + \alpha_{2i-1}]$$

and consider the convex polytope $\Delta_\alpha \subset \mathbf{R}^n$

$$\Delta_\alpha := \begin{cases} I_\alpha \cap (\mathbf{R}_+ \cdot \Xi_n) & \text{when } m = 2n \\ I_\alpha \cap (\mathbf{R}_+ \cdot \Xi_n) \cap \{x_n = |\rho(m)|\} & \text{when } m = 2n-1 \end{cases} .$$

PROPOSITION 6.7. *1) The image of $\partial : {}^m\mathcal{P}^k(\alpha)_+ \longrightarrow \mathbf{R}^n$ is the whole polytope Δ_α.*

2) If $x \in \Delta_\alpha$ is a regular value of ∂, the even-step map e induces, for $m = 3$, a symplectomorphism from the symplectic reduction $T^n \backslash \partial^{-1}(x)$ onto ${}^n\mathcal{P}^k_+(x)$. □

7. REMARKS AND OPEN PROBLEMS

(7.1) Is there an octonionic version of Section 3 ? Alternately, are there $U_1(\mathbf{H})$ bendings in dimension 5 (like the $U_1(\mathbf{C})$ bending flows in dimension 3 and $U_1(R)$ flippings in dimension 2) ?

(7.2) Observe that the inclusion ${}^m\mathcal{P}^k \subset {}^m\mathcal{P}^{k+1}$ becomes a bijection when $k \geq m-1$ (triangles are always planar, etc.). In what ways are these spaces ${}^m\mathcal{P}^{m-1}$ more natural than the unstable ones ?

(7.3) The m-polygons whose first diagonal is of a given length forms a sphere bundle over a space of $(m-1)$-polygons. (For $k = 3$ this is just symplectic reduction by the first bending circle.) This gives an inductive way to construct the space of m-polygons by gluing together (sphere bundles over) the spaces of $(m-1)$-polygons; it would require identification of these sphere bundles, which in $k = 3$ might be done using the Duistermaat-Heckman theorem (where the circle bundle is determined by its Euler class).

Alternately one might work out the fibers of the whole map d of section 5. Unfortunately in dimensions above 3 these are always singular (at, in particular, the planar polygons).

(7.4) In [KM1] and [Wa] there are presented "wall-crossing arguments" for identifying the spaces ${}^m\mathcal{P}^2(\alpha)$. It would be nice to relate these to a combination of [Du] and the paper [GS2], which presents its own wall-crossing arguments for any symplectic reduction by a torus.

(7.5) A space of great interest nowadays is the moduli space of flat $SU(2)$ connections on a punctured Riemann sphere — in the language of this paper, geodesic polygons in S^3 (rather than $\mathbf{R}^3$). The spaces here can be seen as limiting versions where the radius of S^3 goes to infinity. We do not know how to adapt the Gel′fand-MacPherson correspondence to this case; one definite complication is that it is no longer the symmetric group but the braid group which permutes the edges, and that action is not complex.

(7.6) By averaging the Riemannian metric with respect to the bending torus, one can deform the complex structure on a space of prodigal polygons to that of the corresponding toric variety. Is the original complex structure that of a toric variety (not just in the same deformation class)?

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