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3.12. $(3A_2)$. Let $X = (\mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1)^\wedge(p)$ be the blow up of $\mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$ in the point p. The cup form of X is

$$x_4^3 + 6x_1x_2x_3 .$$

3.13. $(2A_1A_2)$. Consider the curve $C = Z(s) \subset \mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$ where $s \in H^0(\mathcal{O}(1,1,0) \oplus \mathcal{O}(0,0,1))$ is a general section, and let X be the blow up of $\mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$ along C. The cup form

$$6x_1x_2x_3 - 3x_1x_4^2 - 3x_2x_4^2 - 2x_4^3$$

of X defines a cubic surface with A_1-singularities in $[1:0:0:0]$ and $[0:1:0:0]$ and an A_2-singularity in $[0:0:1:0]$.

3.14. (D_4^{II}). Let $X := \widehat{\mathbf{P}_1 \times \mathbf{P}}_2(p_1, p_2)$ be the blow up of $\mathbf{P}_1 \times \mathbf{P}_2$ in the points p_1 and p_2. Its cup form is described by the polynomial

$$3x_1x_2^2 + x_3^3 + x_4^3 .$$

This polynomial is the equation of a cubic surface with a D_4-singularity in $[1:0:0:0]$.

3.15. *A Non-Singular Quadric with a Transversal Plane.* Manifolds with such cup forms may be obtained as suitable $\mathbf{P}_1$-bundles over surfaces. Indeed, let Y be a smooth surface with $b_2 = 3$. W. r. t. a suitable basis (h_1, h_2, h_3) of $H^2(Y, \mathbf{Z})$, its cup form is given by $x_1^2 + x_2^2 + x_3^2$. Now, let E be a vector bundle of rank 2 such that $c_1^2(E) - c_2(E) \neq 0$. Let $X := \mathbf{P}(E) \xrightarrow{\pi} Y$ and choose $(\pi^*h_1, \pi^*h_2, \pi^*h_3, c_1(\mathcal{O}_X(1)))$ as a basis of $H^2(X, \mathbf{Z})$. Then, by [OV], Prop. 15, the cup form of X is given by

$$(c_1^2(E) - c_2(E))x_4^3 + x_4(x_1^2 + x_2^2 + x_3^2) .$$

3.16. *A Quadric Cone with a Transversal Plane.* Let Y be a simply connected surface with $b_2 = 3$ and torsion free homology. The cup form of Y is given by a quadratic polynomial $q(x_1, x_2, x_3)$ defining a smooth conic. Thus, the cup form of $Y \times \mathbf{P}_1$ is given by

$$x_4\, q(x_1, x_2, x_3) .$$

4. Real Cubic Forms which are not Cup Forms of Projective Algebraic Manifolds

In the paper [Sch2], the author investigated the restrictions on the real cubic forms of projective manifolds imposed by the so called Hodge-Riemann bilinear relations:

THEOREM 8. *Let X be a Kählerian threefold and $h \in H^2(X, \mathbf{R})$ be a Kähler class. Then the map*

$$\begin{array}{llcl} \langle .,. \rangle : & H^2(X,\mathbf{R}) \times H^2(X,\mathbf{R}) & \longrightarrow & \mathbf{R} \\ & (a,b) & \longmapsto & a \cup b \cup h \end{array}$$

is a non-degenerate, symmetric bilinear form of signature $(2h^{2,0}+1, h^{1,1}-1)$.

One can restate this theorem in such a form as to obtain – at least in theory – some explicit inequalities in the coefficients of cubic polynomials which are satisfied by the cup forms of Kählerian and hence projective algebraic threefolds. The main result of [Sch2] is

THEOREM 9. *For $n \geq 4$, the polynomial*

$$x_0 \left(\frac{4-n}{4} x_0^2 - 3x_1^2 - \cdots - 3x_n^2 \right)$$

cannot occur as the (real) cup form of a projective algebraic threefold with $b_1 = 0$ and $b_3 = 0$.

As a corollary, one obtains the following generalization of a result of Campana and Peternell [CP]:

THEOREM 10. *For $n \geq 4$, twistor spaces over $\sharp_{i=1}^n \mathbf{P}_2$ are not homeomorphic to projective algebraic threefolds.*

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