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4-MANIFOLDS, GROUP INVARIANTS, AND l_2-BETTI NUMBERS

by Beno ECKMANN

It has been known for some time that closed 4-manifolds provide, via the fundamental group and the Euler characteristic, interesting invariants for finitely presented groups. In this short survey we describe these and more refined invariants (using also the signature of the manifold), and explain some of their significance. The invariants are not easily calculated in general, but quite good information is obtained using l_2-Betti numbers.

The topic has been developed by several authors, more or less independently. We mention Hausmann-Weinberger [H-W], Kotschick [K], Lück [L], and myself [E1], [E2]. The paper [K] contains a wealth of information on the invariants and further important references; the application of l_2-Betti numbers appears in [E2] and in [L].

1. A BASIC CONSTRUCTION

1.1. We will always denote by M a connected orientable closed 4-manifold (compact without boundary) admitting a cell decomposition. The fundamental group $G = \pi_1(M)$ is finitely presented. Indeed, homotopy classes of loops can be represented by edge-polygons and null-homotopies of these by using 2-cells. Conversely, any finitely presented group G is the fundamental group of a closed 4-manifold. If

$$G = \langle g_1, \ldots, g_m \mid r_1, \ldots, r_n \rangle$$

is a presentation of G, there is a standard procedure for constructing such a manifold: One first puts $M' = S^1 \times S^3 + \cdots + S^1 \times S^3$, connected sum, one copy for each generator g_i of G. Then $\pi_1(M')$ is a free group on generators $g_1, \ldots, g_m$. A relator, say r_1, is a word in the g_i and can be represented by a loop S^1 in M'.

A tubular neighbourhood $S^1 \times B^3$ of S^1, where B^k is the k-dimensional ball, has boundary $S^1 \times S^2$. Replacing the interior by $B^2 \times S^2$ with the same boundary yields a new 4-manifold where the element corresponding to r_1 has been killed; and similarly for the other r_i. Let M_0 be the 4-manifold thus obtained, fulfilling $\pi_1(M_0) = G$. The idea of that construction can already be found in the old book [S-T]. Much later the procedure, in a more general context, has been called "elementary surgery".

1.2. We recall that the (good old) Euler characteristic $\chi(X)$ of a finite cell complex X is the alternating sum

$$\chi(X) = \sum (-1)^i \alpha_i ,$$

where α_i is the number of i-cells. It is easily computed for M_0 above: For M' it is $2 - 2m$ since it is $= 0$ for $S^1 \times S^3$ and since it decreases by 2 in a connected sum. Under the surgery process above it increases by 2 [use the fact that for the union of two complexes X and Y with intersection Z the characteristic is $\chi(X) + \chi(Y) - \chi(Z)$; and that $\chi(B^2 \times S^2) = 2$]. Whence

$$\chi(M_0) = 2 - 2m + 2n = 2 - 2(m - n) .$$

The difference $m - n$ is called the deficiency of the presentation of G.

1.3. On the other hand the characteristic can be expressed by the Betti numbers of the cell complex X as $\sum (-1)^i \beta_i(X)$ where $\beta_i(X) = \dim_{\mathbf{R}} H_i(X; \mathbf{R})$ (and is therefore a topological invariant). Moreover the β_i of a manifold fulfill Poincaré duality, i.e. they are equal in complementary dimensions. Thus $\chi(M) = 2 - 2\beta_1(M) + \beta_2(M)$. We recall that homology in dimension 1 depends on the fundamental group G only; β_1 is the $\mathbf{Q}$-rank of G Abelianised and we write $\beta_1(G)$ for $\beta_1(M)$. Comparing with $\chi(M_0)$ above we see that the deficiency of the presentation is $\leq \beta_1(G)$. Thus there is a maximum for the deficiency of all presentations of G, called the deficiency $\mathrm{def}(G)$ of G. [For this simple side result there are, of course, much easier arguments.]

2. The Hausmann-Weinberger invariant

2.1. As seen above, the Euler characteristic of a 4-manifold M with given finitely presented fundamental group G is bounded below by $2 - 2\beta_1(G)$. The minimum of $\chi(M)$ for all such M has been considered by Hausmann-Weinberger [H-W] and denoted by $q(G)$. Using M_0 above we have the inequalities

$$2 - 2\beta_1(G) \leq q(G) \leq 2 - 2\,\mathrm{def}(G) .$$