Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 43 (1997)

Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THEOREM OF INGHAM IMPLYING THAT DIRICHLET'S L-FUNCTIONS HAVE NO ZEROS WITH REAL PART ONE

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Kapitel: §1. Introduction

DOI: https://doi.org/10.5169/seals-63280

L'Enseignement Mathématique, t. 43 (1997), p. 281–284

A THEOREM OF INGHAM IMPLYING THAT DIRICHLET'S L-FUNCTIONS HAVE NO ZEROS WITH REAL PART ONE

by Paul T. BATEMAN

§ 1. INTRODUCTION

Using Landau's lemma on Dirichlet series with non-negative coefficients, A. E. Ingham in [1] proved the following theorem.

INGHAM'S THEOREM. *Let*

$$g(s) = g(s, \epsilon) = \prod_p \left(1 - \frac{\epsilon(p)}{p^s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{\epsilon(n)}{n^s} \qquad (\mathrm{Re}\, s > 1),$$

where the product is extended over all prime numbers p *and* ϵ *is a bounded completely multiplicative arithmetic function. Suppose that* g *can be analytically continued into some domain containing the closed interval* $\left[\frac{1}{2}, 1\right]$ *of the real axis. Then*

$$g(1) \neq 0 .$$

We recall that an arithmetic function ϵ is said to be completely multiplicative if $\epsilon(mn) = \epsilon(m)\,\epsilon(n)$ for all positive integers m and n. Since a completely multiplicative arithmetic function ϵ is determined by the values $\epsilon(p)$ for primes p, it is immediate that ϵ is bounded if and only if $|\epsilon(p)| \leq 1$ for all primes p. Actually Ingham assumed that $|\epsilon(p)|$ is either 0 or 1 for any prime p, but his proof can easily be modified to require only that $|\epsilon(p)| \leq 1$. (Cf. [6]).

The most interesting application of Ingham's theorem is that obtained by taking $\epsilon(n) = \chi(n)n^{-i\alpha}$, where χ is a residue character modulo k and α is a real number which is different from zero if χ is the principal character. Then

$g(s) = L(s + i\alpha, \chi)$ and the theorem gives the assertion $L(1 + i\alpha, \chi) \neq 0$. The main interest in Ingham's theorem is its breadth of applicability. In contrast the familiar use of a trigonometric inequality does not cover the case in which $\alpha = 0$ and χ is a real non-principal character; that case is the very one to which Landau's lemma applies most easily (cf. §2 of [3]).

Ingham proved the theorem by establishing and using the identity

$$(*) \qquad \zeta(s)\, g(s, \epsilon)\, g(s, \eta)\, g(s, \epsilon\eta) = g(2s, \epsilon\eta) \sum_{n=1}^{\infty} E(n)\, H(n)\, n^{-s},$$

where ζ denotes the Riemann zeta function, η is another completely multiplicative arithmetic function, $\epsilon\eta$ is the pointwise product of ϵ and η, and

$$E(n) = \sum_{d|n} \epsilon(d), \quad H(n) = \sum_{d|n} \eta(d).$$

This is a generalization of a result of Ramanujan for the case $\epsilon(n) = n^a$, $\eta(n) = n^b$, where a and b are fixed complex numbers.

While the identity $(*)$ is of some interest in itself, we show that for the purpose of proving Ingham's theorem there is no reason to make the detour needed to prove $(*)$. Of course we still require Landau's lemma, which we state in the following form.

LANDAU'S LEMMA. *Suppose β and γ are real numbers with $\beta < \gamma$. Suppose that $c_n \geq 0$ for $n = 1, 2, 3, \ldots$ and that the series $\sum c_n n^{-s}$ converges for* $\operatorname{Re} s > \gamma$. *Put*

$$f(s) = \sum_{n=1}^{\infty} c_n\, n^{-s} \qquad (\operatorname{Re} s > \gamma).$$

If f can be analytically continued into some domain containing the closed interval $[\beta, \gamma]$ of the real axis, then the series $\sum c_n n^{-\beta}$ converges.

For a proof of Landau's lemma see [5], [4], or §2 of [2].

The proof of Ingham's theorem which we give here uses an argument similar to that used in [4] and [5], except that the argument in those two papers was phrased in such a way as to require analytic continuation into a domain containing the interval $[0, 1]$ of the real axis instead of the interval $\left[\frac{1}{2}, 1\right]$. The interval $\left[\frac{1}{2}, 1\right]$ could not be replaced in the hypothesis of Ingham's theorem by a shorter interval, i.e., one of the form $[\theta, 1]$, where $\theta > \frac{1}{2}$; this is shown by the example in which ϵ is the Liouville function λ and $g(s) = \zeta(2s)/\zeta(s) = \sum \lambda(n)\, n^{-s}$.