

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 43 (1997)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THEOREM OF INGHAM IMPLYING THAT DIRICHLET'S L-FUNCTIONS HAVE NO ZEROS WITH REAL PART ONE
Autor: Bateman, Paul T.
Kapitel: §1. Introduction
DOI: <https://doi.org/10.5169/seals-63280>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

Download PDF: 01.04.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

A THEOREM OF INGHAM
IMPLYING THAT DIRICHLET'S L -FUNCTIONS
HAVE NO ZEROS WITH REAL PART ONE

by Paul T. BATEMAN

§ 1. INTRODUCTION

Using Landau's lemma on Dirichlet series with non-negative coefficients, A. E. Ingham in [1] proved the following theorem.

INGHAM'S THEOREM. *Let*

$$g(s) = g(s, \epsilon) = \prod_p \left(1 - \frac{\epsilon(p)}{p^s} \right)^{-1} = \sum_{n=1}^{\infty} \frac{\epsilon(n)}{n^s} \quad (\operatorname{Re} s > 1),$$

where the product is extended over all prime numbers p and ϵ is a bounded completely multiplicative arithmetic function. Suppose that g can be analytically continued into some domain containing the closed interval $[\frac{1}{2}, 1]$ of the real axis. Then

$$g(1) \neq 0.$$

We recall that an arithmetic function ϵ is said to be completely multiplicative if $\epsilon(mn) = \epsilon(m)\epsilon(n)$ for all positive integers m and n . Since a completely multiplicative arithmetic function ϵ is determined by the values $\epsilon(p)$ for primes p , it is immediate that ϵ is bounded if and only if $|\epsilon(p)| \leq 1$ for all primes p . Actually Ingham assumed that $|\epsilon(p)|$ is either 0 or 1 for any prime p , but his proof can easily be modified to require only that $|\epsilon(p)| \leq 1$. (Cf. [6]).

The most interesting application of Ingham's theorem is that obtained by taking $\epsilon(n) = \chi(n)n^{-i\alpha}$, where χ is a residue character modulo k and α is a real number which is different from zero if χ is the principal character. Then

$g(s) = L(s + i\alpha, \chi)$ and the theorem gives the assertion $L(1 + i\alpha, \chi) \neq 0$. The main interest in Ingham's theorem is its breadth of applicability. In contrast the familiar use of a trigonometric inequality does not cover the case in which $\alpha = 0$ and χ is a real non-principal character; that case is the very one to which Landau's lemma applies most easily (cf. §2 of [3]).

Ingham proved the theorem by establishing and using the identity

$$(*) \quad \zeta(s) g(s, \epsilon) g(s, \eta) g(s, \epsilon\eta) = g(2s, \epsilon\eta) \sum_{n=1}^{\infty} E(n) H(n) n^{-s},$$

where ζ denotes the Riemann zeta function, η is another completely multiplicative arithmetic function, $\epsilon\eta$ is the pointwise product of ϵ and η , and

$$E(n) = \sum_{d|n} \epsilon(d), \quad H(n) = \sum_{d|n} \eta(d).$$

This is a generalization of a result of Ramanujan for the case $\epsilon(n) = n^a$, $\eta(n) = n^b$, where a and b are fixed complex numbers.

While the identity (*) is of some interest in itself, we show that for the purpose of proving Ingham's theorem there is no reason to make the detour needed to prove (*). Of course we still require Landau's lemma, which we state in the following form.

LANDAU'S LEMMA. *Suppose β and γ are real numbers with $\beta < \gamma$. Suppose that $c_n \geq 0$ for $n = 1, 2, 3, \dots$ and that the series $\sum c_n n^{-s}$ converges for $\operatorname{Re} s > \gamma$. Put*

$$f(s) = \sum_{n=1}^{\infty} c_n n^{-s} \quad (\operatorname{Re} s > \gamma).$$

If f can be analytically continued into some domain containing the closed interval $[\beta, \gamma]$ of the real axis, then the series $\sum c_n n^{-\beta}$ converges.

For a proof of Landau's lemma see [5], [4], or §2 of [2].

The proof of Ingham's theorem which we give here uses an argument similar to that used in [4] and [5], except that the argument in those two papers was phrased in such a way as to require analytic continuation into a domain containing the interval $[0, 1]$ of the real axis instead of the interval $[\frac{1}{2}, 1]$. The interval $[\frac{1}{2}, 1]$ could not be replaced in the hypothesis of Ingham's theorem by a shorter interval, i.e., one of the form $[\theta, 1]$, where $\theta > \frac{1}{2}$; this is shown by the example in which ϵ is the Liouville function λ and $g(s) = \zeta(2s)/\zeta(s) = \sum \lambda(n) n^{-s}$.