

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 43 (1997)  
**Heft:** 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** THEOREM OF INGHAM IMPLYING THAT DIRICHLET'S L-FUNCTIONS HAVE NO ZEROS WITH REAL PART ONE  
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**Kapitel:** §2. Proof of Ingham's Theorem  
**DOI:** <https://doi.org/10.5169/seals-63280>

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## §2. PROOF OF INGHAM'S THEOREM

Suppose that  $g(1) = 0$ . We show that this assumption leads to a contradiction. We consider the function

$$F(s) = \zeta(s)^2 g(s) g^*(s) \quad (\operatorname{Re} s > 1),$$

where  $g^*(s) = g(s, \bar{\epsilon})$  and  $\bar{\epsilon}$  is the arithmetic function which is the complex conjugate of  $\epsilon$ . Clearly  $g^*(1) = \overline{g(1)} = 0$ . By hypothesis  $g$  is regular along the stretch  $[\frac{1}{2}, 1]$  of the real axis and so therefore is  $g^*$ , since  $g^*(s) = \overline{g(\bar{s})}$ . Hence  $F$  is regular on  $[\frac{1}{2}, 1]$ , since the double pole of  $\zeta^2$  at  $s = 1$  is canceled by the zeros of  $g$  and  $g^*$  there.

Using the identity

$$(1 - z)^{-1} = \exp\left(\sum_{k=1}^{\infty} z^k/k\right) \quad (|z| < 1),$$

we obtain (for  $\operatorname{Re} s > 1$ )

$$\begin{aligned} F(s) &= \prod_p \left(1 - \frac{1}{p^s}\right)^{-2} \left(1 - \frac{\epsilon(p)}{p^s}\right)^{-1} \left(1 - \frac{\bar{\epsilon}(p)}{p^s}\right)^{-1} \\ &= \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{2 + \epsilon(p)^k + \bar{\epsilon}(p)^k}{kp^{ks}}\right) \\ &= \prod_p \left\{1 + \left(\sum_{k=1}^{\infty} \frac{2 + \epsilon(p)^k + \bar{\epsilon}(p)^k}{kp^{ks}}\right) + \frac{1}{2} \left(\sum_{k=1}^{\infty} \frac{2 + \epsilon(p)^k + \bar{\epsilon}(p)^k}{kp^{ks}}\right)^2 + \dots\right\}. \end{aligned}$$

Thus  $F$  has a Dirichlet series expansion

$$F(s) = \sum_{n=1}^{\infty} a(n) n^{-s} \quad (\operatorname{Re} s > 1).$$

Furthermore, since

$$2 + \epsilon(p)^k + \bar{\epsilon}(p)^k = 2 + 2 \operatorname{Re}\{\epsilon(p)^k\} \geq 0,$$

we have  $a(n) \geq 0$  for all  $n$ .

At this point we deviate from the approach used in [4] and [5] by noting that  $a(p^2) \geq 1$  for each prime  $p$ . For, since

$$F(s) = \prod_p \left(1 + \frac{2}{p^s} + \frac{3}{p^{2s}} + \dots\right) \left(1 + \frac{\epsilon(p)}{p^s} + \frac{\epsilon(p)^2}{p^{2s}} + \dots\right) \left(1 + \frac{\bar{\epsilon}(p)}{p^s} + \frac{\bar{\epsilon}(p)^2}{p^{2s}} + \dots\right),$$

we find that

$$\begin{aligned}
a(p^2) &= 3 + 2\epsilon(p) + 2\bar{\epsilon}(p) + \epsilon(p)^2 + \epsilon(p)\bar{\epsilon}(p) + \bar{\epsilon}(p)^2 \\
&= 2 - \epsilon(p)\bar{\epsilon}(p) + \{1 + \epsilon(p) + \bar{\epsilon}(p)\}^2 \\
&\geq 2 - |\epsilon(p)|^2 \geq 1.
\end{aligned}$$

Thus

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^{1/2}} \geq \sum_p \frac{a(p^2)}{p} \geq \sum_p \frac{1}{p}.$$

In view of the divergence of  $\sum p^{-1}$ , it follows that  $\sum a(n)n^{-1/2}$  diverges.

On the other hand, applying Landau's lemma with  $c_n = a(n)$ ,  $\beta = \frac{1}{2}$ ,  $\gamma = 1$ , we find that  $\sum a(n)n^{-1/2}$  converges. This contradiction shows that the assumption  $g(1) = 0$  is untenable and so the proof is complete.

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(Reçu le 16 avril 1997)

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