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Artikel: THEOREM OF INGHAM IMPLYING THAT DIRICHLET'S L-FUNCTIONS HAVE NO ZEROS WITH REAL PART ONE

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§2. Proof of Ingham's Theorem

Suppose that $g(1) = 0$. We show that this assumption leads to a contradiction. We consider the function

$$F(s) = \zeta(s)^2 g(s) g^*(s) \qquad (\operatorname{Re} s > 1),$$

where $g^*(s) = g(s, \bar{\epsilon})$ and $\bar{\epsilon}$ is the arithmetic function which is the complex conjugate of ϵ. Clearly $g^*(1) = \overline{g(1)} = 0$. By hypothesis g is regular along the stretch $\left[\frac{1}{2}, 1\right]$ of the real axis and so therefore is g^*, since $g^*(s) = \overline{g(\bar{s})}$. Hence F is regular on $\left[\frac{1}{2}, 1\right]$, since the double pole of ζ^2 at $s = 1$ is canceled by the zeros of g and g^* there.

Using the identity

$$(1 - z)^{-1} = \exp\left(\sum_{k=1}^{\infty} z^k / k\right) \qquad (|z| < 1),$$

we obtain (for $\operatorname{Re} s > 1$)

$$\begin{aligned} F(s) &= \prod_p \left(1 - \frac{1}{p^s}\right)^{-2} \left(1 - \frac{\epsilon(p)}{p^s}\right)^{-1} \left(1 - \frac{\bar{\epsilon}(p)}{p^s}\right)^{-1} \\ &= \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{2 + \epsilon(p)^k + \bar{\epsilon}(p)^k}{k p^{ks}}\right) \\ &= \prod_p \left\{1 + \left(\sum_{k=1}^{\infty} \frac{2 + \epsilon(p)^k + \bar{\epsilon}(p)^k}{k p^{ks}}\right) + \frac{1}{2}\left(\sum_{k=1}^{\infty} \frac{2 + \epsilon(p)^k + \bar{\epsilon}(p)^k}{k p^{ks}}\right)^2 + \cdots\right\}. \end{aligned}$$

Thus F has a Dirichlet series expansion

$$F(s) = \sum_{n=1}^{\infty} a(n) n^{-s} \qquad (\operatorname{Re} s > 1).$$

Furthermore, since

$$2 + \epsilon(p)^k + \bar{\epsilon}(p)^k = 2 + 2\operatorname{Re}\{\epsilon(p)^k\} \geq 0,$$

we have $a(n) \geq 0$ for all n.

At this point we deviate from the approach used in [4] and [5] by noting that $a(p^2) \geq 1$ for each prime p. For, since

$$F(s) = \prod_p \left(1 + \frac{2}{p^s} + \frac{3}{p^{2s}} + \cdots\right)\left(1 + \frac{\epsilon(p)}{p^s} + \frac{\epsilon(p)^2}{p^{2s}} + \cdots\right)\left(1 + \frac{\bar{\epsilon}(p)}{p^s} + \frac{\bar{\epsilon}(p)^2}{p^{2s}} + \cdots\right),$$

we find that

$$\begin{aligned} a(p^2) &= 3 + 2\epsilon(p) + 2\overline{\epsilon}(p) + \epsilon(p)^2 + \epsilon(p)\,\overline{\epsilon}(p) + \overline{\epsilon}(p)^2 \\ &= 2 - \epsilon(p)\,\overline{\epsilon}(p) + \{1 + \epsilon(p) + \overline{\epsilon}(p)\}^2 \\ &\geq 2 - |\epsilon(p)|^2 \geq 1\,. \end{aligned}$$

Thus

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^{1/2}} \geq \sum_{p} \frac{a(p^2)}{p} \geq \sum_{p} \frac{1}{p}\,.$$

In view of the divergence of $\sum p^{-1}$, it follows that $\sum a(n)\,n^{-1/2}$ diverges.

On the other hand, applying Landau's lemma with $c_n = a(n)$, $\beta = \frac{1}{2}$, $\gamma = 1$, we find that $\sum a(n)\,n^{-1/2}$ converges. This contradiction shows that the assumption $g(1) = 0$ is untenable and so the proof is complete.

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