

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 43 (1997)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON CYCLOTOMIC POLYNOMIALS, POWER RESIDUES, AND RECIPROCITY LAWS
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Kapitel: 5. Homogeneous polynomials
DOI: <https://doi.org/10.5169/seals-63283>

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5. HOMOGENEOUS POLYNOMIALS

Generalizing Theorem 1 to include homogeneous polynomials introduces subtle difficulties, which we address in the following proof.

Proof of Theorem 2. Let K and L be as in the proof of Theorem 1. We set $\pi_n = y - qx\zeta_n$ and $\pi = N_{L/K}(\pi_n)$ and note $p = N_K(\pi)$ with the assumption $s > 0$. Now let a be an integer dividing x , and decompose a as $a = a'q^k$ where a' is not divisible by q . Let $y' = N_{L/K}(y) = y^{[L:K]}$. We remark that $\pi \equiv y' \pmod{qa}$ since $\pi_n \equiv y \pmod{qa}$.

We can now apply reciprocity. We are interested in evaluating the symbol $(a/\pi)_l$. We can use reciprocity laws (5) and (6) along with multiplicativity from Theorems 5(a) and 6(a) in the following manner. We have

$$\left(\frac{a}{\pi}\right)_l = \left(\frac{a'}{\pi}\right)_l \left(\frac{q}{\pi}\right)_l^k = \left(\frac{\pi}{a'}\right)_l (\pi, a')_l (\pi, q)_l^k = \left(\frac{\pi}{a'}\right)_l (\pi, a)_l.$$

Now $\pi \equiv y' \pmod{a'}$ so $(\pi/a')_l = (y'/a')_l$ by Theorem 5(c). And letting $(y')^{-1}$ denote the q -adic inverse of y' in \mathbf{Z}_q , we have

$$(\pi, a)_l = (\pi(y')^{-1}, a)_l (y', a)_l.$$

Let $\pi' = \pi(y')^{-1}$. Now $\pi' \equiv 1 \pmod{qa}$. So $\pi' \equiv 1 \pmod{q}$, and if q divides a , then $\pi' \equiv 1 \pmod{q^2}$. Thus the fact that λ_q^2 divides q implies $\pi' \equiv 1 \pmod{f_l(a)}$ by Corollary 8. Thus $(\pi', a)_l = 1$.

We now have that

$$\left(\frac{a}{\pi}\right)_l = \left(\frac{y'}{a'}\right)_l (y', a)_l.$$

The symbol $(y'/a')_l$ is an l -th root of unity, and by Theorem 5(d) an element $\sigma \in G_K$ acts on it as follows:

$$\sigma\left(\frac{y'}{a'}\right)_l = \left(\frac{\sigma y'}{\sigma a'}\right)_l = \left(\frac{y'}{a'}\right)_l,$$

since a' and y' are rational integers. Since l is odd, the only such root of unity fixed under the action of the Galois group is 1. In the same manner, Theorem 6(h) enables us to see that $(y', a)_l = 1$. We therefore conclude that $(a/\pi)_l = 1$. \square