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6. WHAT ABOUT q ?

We pose the question: is q necessarily a q^s th power modulo p in Theorem 2? A numerical test will quickly show that this is most certainly not always so. However, Theorem 3 tells us that the answer is in fact “yes” in most cases.

Proof of Theorem 3. Let K, L, π be as in the proof of Theorem 2 and assume $s > 0$. Note first that $(q/\pi)_l = (\pi, q)_l$ by reciprocity law (6). We will evaluate the latter symbol.

Viewing $\Phi_{n/l}(X, Y)$ as a polynomial over K , we have that

$$\pi = N_{L/K}(y - qx\zeta_n) = \Phi_{n/l}(y, qx\zeta_l),$$

where ζ_l in this equation is given by $\zeta_l^{n/l} = \zeta_n^{n/l}$. We now state the following generating formula for homogeneous cyclotomic polynomials:

$$(8) \quad \Phi_m(X, Y) = \frac{X^m - Y^m}{\prod_{\substack{d|m \\ 0 < d < m}} \Phi_d(X, Y)}.$$

Applying this formula recursively, we see that π is expressible as a product of numbers of the form $y^r - (qx\zeta_l)^r$ and reciprocals of such numbers, where r is some positive divisor of n/l . To show that $(\pi, q)_l = 1$, it is by bimultiplicativity enough to show that $(y^r - (qx\zeta_l)^r, q)_l = 1$ for all such r . And since n/l is relatively prime to q , it will clearly suffice to show that $(y - qx\zeta_l, q)_l = 1$ for any choice of ζ_l and integers x and y with y relatively prime to q .

We have

$$(y - qx\zeta_l, q)_l = (y, q)_l(1 - qxy^{-1}\zeta_l, q)_l.$$

The first symbol $(y, q)_l$ is fixed under the action of the Galois group $G_{\mathbf{Q}_q(\zeta_l)/\mathbf{Q}_q}$ by Theorem 6(h) since $y, q \in \mathbf{Q}_q$. As an l th root of unity with l odd, it must therefore be 1.

By Theorem 6(f), $(1 - qxy^{-1}\zeta_l, qxy^{-1}\zeta_l)_l = 1$. But by bimultiplicativity, this means that

$$(1 - qxy^{-1}\zeta_l, q)_l = (1 - qxy^{-1}\zeta_l, xy^{-1})_l^{-1} (1 - qxy^{-1}\zeta_l, \zeta_l)_l^{-1}.$$

Corollary 8 yields that $1 - qxy^{-1}\zeta_l \equiv 1 \pmod{f_l(xy^{-1})}$, and so the first symbol on the right is 1. The second symbol can be evaluated by turning it back into a power residue symbol and applying (4). Since ζ_l is a unit in the ring of integers of K , the reciprocity law (5) yields

$$(9) \quad (1 - qxy^{-1}\zeta_l, \zeta_l)_l = \left(\frac{\zeta_l}{1 - qxy^{-1}\zeta_l} \right)_l = \zeta_l^{(N_K(1 - qxy^{-1}\zeta_l) - 1)/l}.$$

Thus $(1 - qxy^{-1}\zeta_l, \zeta_l)_l$ will equal 1 if and only if $N_K(1 - qxy^{-1}\zeta_l) \equiv 1 \pmod{q^{2s}}$. In fact,

$$N_K(1 - qxy^{-1}\zeta_l) = \sum_{i=0}^{q-1} (qxy^{-1})^{iq^{s-1}} \equiv 1 \pmod{q^{q^{s-1}}}.$$

It is easily seen that $q^{s-1} \geq 2s$ exactly when stated in the theorem. \square

One remark on the case $s = 1$. If in fact we take $n = q$, then since $\Phi_q(X) = 1 + X + \dots + X^{q-1}$ we have that $p \equiv 1 \pmod{q^2}$ if and only if q divides x . Then q is a q th power modulo p if and only if x is divisible by q , in stark contrast to the above theorem.

7. THE EVEN CASE

We now turn to the case of $q = 2$. Given a positive integer s , let us set $l = 2^s$. We refrain from proving the theorem for the more general case of homogeneous polynomials, though it holds under such a generalization.

Any $\alpha \in \mathbf{Q}_2^*$ may be written uniquely as $\alpha = \xi 2^b (-3)^c$ where $\xi = \pm 1$, $b \in \mathbf{Z}$, and $c \in \mathbf{Z}_2$. Note that $b = v_2(\alpha)$, where v_2 is the 2-adic valuation. Denote by $f_l(\alpha)$ the conductor of the norm residue character $(\cdot, \alpha)_l$ in $\mathbf{Q}_2(\zeta_l)$. The conductors in this case have been worked out by Despina Prapavessi in [P]. We use a corrected version of her theorem [Sh1].

THEOREM 9 (Prapavessi). *Let $\alpha \in \mathbf{Q}_2^*$ and write $\alpha = \xi 2^b (-3)^c$ as above. Let $w = \min \{v_2(b), v_2(c) + 2\}$. Then if $\xi = 1$,*

$$f_l(\alpha) = \begin{cases} (8) & \text{if } w = 0, \\ (4) & \text{if } w = 1 \text{ and } s \geq 2, \\ (\lambda_{2^{w-1}}) & \text{if } 2 \leq w \leq s \text{ and } w = v_2(c) + 2, \\ (\lambda_{2^w} \lambda_{2^{w+1}}) & \text{if } 2 \leq w < s - 1 \text{ and } w \leq v_2(c) + 1, \\ (\lambda_{2^{s-1}}) & \text{if } 2 \leq w = s - 1 \text{ and } w = v_2(c) + 1, \\ (1) & \text{otherwise.} \end{cases}$$