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Artikel: AMENABILITY AND GROWTH OF ONE-RELATOR GROUPS
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Denoting now by

$$\lambda_*(G) = \inf_A \lambda_A(G)$$

the *minimal growth rate* of G, where the infimum is taken over all finite generating systems, the group G has *uniform exponential growth* if $\lambda_*(G) > 1$. This last concept is due to Avez [A] where the number

$$h(G) = \log(\lambda_*(G))$$

is called the *entropy* of the group G and it is discussed in [GrLP], [SW] and in the survey paper [GH].

The simplest example of a group with uniformly exponential growth is the free group $\mathbf{F}_m$ of finite rank $m \geq 2$ for which the minimal growth rate is $\lambda_*(\mathbf{F}_m) = 2m - 1$, see for instance [GH].

It is not known whether a group of exponential growth has necessarily uniformly exponential growth or not. We formulate the following:

0.1. CONJECTURE. *All one-relator groups of exponential growth have uniformly exponential growth.*

Conjecture 0.1 is true for one-relator groups of rank $m \geq 3$ and for one-relator groups with torsion, therefore we focus our attention on two-generated one-relator groups and give sufficient conditions for such groups to have uniformly exponential growth. We present a new method for estimating the minimal growth rate of a finitely generated group using growth functions of the corresponding graded Lie algebra and apply it to one-relator groups.

1. AN ALGORITHM FOR CHECKING AMENABILITY

Let G be a one-relator group with presentation $(*)$; the number m of the generators of G in the presentation is called the rank of the presentation. Untill Section 4 we shall assume that R is cyclically reduced and non trivial.

The next observation is well known. We shall include the proof stressing the algorithmic aspect of the statement.

1.1. LEMMA. *Let $G = \langle a, b, \dots : R(a, b, \dots)\rangle$ be a one-relator group with at least two generators. Then G has a presentation $\langle t, \dots : R'(t, \dots)\rangle$ with $\sigma_t(R') = 0$, where $\sigma_t(R')$ denotes the sum of the exponents of t in the word R'. This second presentation can in fact be produced, starting from the original one, in an algorithmical way.*

Proof. Let a and b be two generators involved in R; if $\sigma_a(R) = 0$ or $\sigma_b(R) = 0$ we are already done. If not, suppose that $0 < |\sigma_a(R)| \leq |\sigma_b(R)|$; by exchanging a with a^{-1} and/or b with b^{-1} if necessary, we can suppose that $0 < \sigma_a(R) \leq \sigma_b(R)$. Set $a' = ab$ and $b' = b$; then, if $R'(a', b')$ is the expression of R in terms of the new generators a' and b', one has $\sigma_{a'}(R') = \sigma_a(R)$ and $|\sigma_{b'}(R')| < \sigma_b(R)$. Applying this procedure inductively for at most $|\sigma_a(R)| + |\sigma_b(R)|$ times one gets the claimed presentation. $\square$

Note that the rank of the second presentation in the previous lemma coincides with the rank of the initial one.

1.2. THEOREM. *The following is an algorithm which establishes if a given one-relator group G with presentation $(*)$ is amenable or not:*

Step 1: If $m \geq 3$ then G is not amenable. If $m = 1$ then G is amenable; if $m = 2$ go to next step.

Step 2: Check if R is a power of one of the generators. If this is the case and the power is proper then G is not amenable, if R coincides, up to inversion, with one of the generators then G is amenable. Otherwise go to next step.

Step 3: Using the algorithm from the above lemma, change the presentation of G so that the sum of the exponents of one of the generators in the relator is zero. Then G is amenable iff, up to a relabeling and inversion of the generators, and up to a cyclic permutation of the relator, the presentation obtained is of the form $\langle t, s : tst^{-1}s^{-n} = 1\rangle$, with $n \in \mathbf{Z} \setminus \{0\}$.

Proof. Recall that the Freiheitssatz of Wilhelm Magnus ([MKS: Thm. 4.10] and [LS: IV Thm. 5.1]) states that, if $R = R(a_1, a_2, \ldots, a_m)$ is a cyclically reduced word in $a_1, a_2, \ldots, a_m$ and involves a_m, then the subgroup of $G = \langle a_1, a_2, \ldots, a_m : R(a_1, a_2, \ldots, a_m) = 1\rangle$ generated by $a_1, a_2, \ldots, a_{m-1}$ is freely generated by them.

(1) If $m \geq 3$ then, by Magnus' Theorem, G contains the free group on two generators and thus it is not amenable. If $m = 1$ then $G = \langle a : a^n = 1\rangle$ is cyclic and therefore amenable.

(2) Let $m = 2$. If R is a proper power of one of the generators, say $R = a^n$ with $|n| \geq 2$, then G is isomorphic to the free product $\mathbf{Z} * \mathbf{Z}_{|n|}$ of the infinite cyclic group and the cyclic group of order $|n| \geq 2$ and it is not amenable because its commutator subgroup is a free group of infinite rank. If R coincides, up to inversion, with one of the generators then G is infinite cyclic and therefore amenable.

(3) Suppose now that $\langle a, b : R(a,b) = 1\rangle$ is a presentation of G with $\sigma_a(R) = 0$. If we denote by $b_i = a^i b a^{-i}$, $i \in \mathbf{Z}$, then the relator R can be expressed as a word in the b_i's just replacing each b^k in $R(a,b)$ by b_j^k, where j is the sum of the exponents of a in the subword of R preceding the given occurrence of b^k. We shall denote this word by $R'(b_m, b_{m+1}, \dots, b_M)$, where m and M are the minimum and, respectively, the maximum subscript occurring in the expression of R'. Note that since $R(a,b)$ is cyclically reduced, then R' is cyclically reduced as well and $m < M$.

It is known [LS: IV, proof of Thm. 5.1] that any one-relator group with ≥ 2 generators is an HNN-extension $(H; A, B, \phi)$ of another one-relator group H. In our situation

$$\begin{aligned} H &= \langle b_m, b_{m+1}, \dots, b_M; R'(b_m, b_{m+1}, \dots, b_M)\rangle \\ A &= \text{ subgroup of } H \text{ generated by } b_m, b_{m+1}, \dots, b_{M-1} \\ B &= \text{ subgroup of } H \text{ generated by } b_{m+1}, b_{m+2}, \dots, b_M \\ \phi &: A \ni b_i \longmapsto b_{i+1} \in B, \quad i = m, m+1, \dots, M-1\,. \end{aligned}$$

Therefore G also admits the following presentation

$$G = \left\langle a, b_m, \dots, b_M : R'(b_m, \dots, b_M) = 1,\ a b_i a^{-1} = b_{i+1},\ i = m, \dots, M-1 \right\rangle.$$

The subgroups A and B are free of rank $M - m$ and if $M - m \geq 2$ then G is not amenable.

Suppose now that $M - m = 1$, so that $A = \langle b_m\rangle \cong B = \langle b_M\rangle \cong \mathbf{Z}$. It is known ([H: Prop. 3.3]) that an HNN-extension $(H; A, B, \phi)$, such that A and B are both proper subgroups of the base group H, contains the free group $\mathbf{F}_2$. Thus, if $A \neq H \neq B$, then G is non amenable.

Suppose that $A = H$ (the case $B = H$ is similar). Then $H = \langle b_m\rangle \cong \mathbf{Z}$ and $b_M = b_m^k$ for a suitable $k \in \mathbf{Z} \setminus \{0\}$. Replacing a by t and b_m by s in the above presentation for G, one gets the presentation

$$G = \left\langle t, s : tst^{-1} = s^k \right\rangle$$

of type 3_b. from the list $(**)$ and so G is amenable. □

1.3. COROLLARY. *For amenable one-relator groups the isomorphism problem is solvable.*

Proof. Suppose two one-relator groups which are amenable are given. Then, in the algorithmical way described above, one gets two presentations from the list $(**)$ and the procedure of recognition becomes obvious since any two groups from the list with different presentations are in fact non-isomorphic. □