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Artikel: AMENABILITY AND GROWTH OF ONE-RELATOR GROUPS
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3.6. THEOREM ([L 1,2]). *Let R be an element of the free group F of finite rank m which is primitive with respect to the lower central series. Denote by $k = \omega(R)$ its weight and by $\langle R \rangle$ the normal closure of R in F. Let $G = F/\langle R \rangle$ and let $\mathcal{L}(F)$ and $\mathcal{L}(G)$ be the corresponding Lie algebras. Let then r be the image of R in $\mathcal{L}_k(F)$, the k-th component of $\mathcal{L}(F)$ and denote by I the ideal of $\mathcal{L}(F)$ generated by r.*

Then I is the kernel of the canonical homomorphism of $\mathcal{L}(F)$ onto $\mathcal{L}(G)$, i.e.

$$\mathcal{L}(G) = \mathcal{L}(F)/I.$$

Moreover for all $n \geq 1$ the abelian group $\mathcal{L}_n(G)$ is a torsion free group whose rank is the n-th coefficient of the Maclaurin power series of the function

$$U(z) = \frac{1}{1 - mz + z^k}.$$

4. MORE ON UNIFORMLY EXPONENTIAL GROWTH OF ONE-RELATOR GROUPS

Any two-generated one-relator group G can be presented in the form $G = \langle a, b : a^k w(a,b) = 1 \rangle$ where $k \in \mathbf{Z}$ and $w(a,b)$ belongs to the commutator subgroup $[F,F]$ of the free group $F = F(a,b)$ freely generated by a and b (this follows from Lemma 1.1). Since a and b constitute a basis in $F/\gamma_2(F)$ and $[a,b]$ generates $\gamma_2(F)/\gamma_3(F)$, one can also present G in the form

$$G = \langle a, b : a^k [a,b]^l w(a,b) = 1 \rangle$$

where $k, l \in \mathbf{Z}$ and $w(a,b) \in \gamma_3(F)$.

In this section we shall see that, under suitable assumptions on k, l and $w(a,b)$, the corresponding group has uniformly exponential growth.

As an application of Labute's Theorem we get the following:

4.1. PROPOSITION. *Let $G = \langle a, b : R(a,b) = 1 \rangle$ be such that R is primitive with respect to $\{\gamma_n(F)\}_{n=1}^{\infty}$ and $R \in \gamma_3(F)$. Then G has uniformly exponential growth.*

Proof. If $\omega(R) \geq 3$, Theorem 3.6 shows that the corresponding function $U(z)$ has a pole z_0 with $0 < z_0 < 1$. It follows that the coefficients $c_n(G)$ grow exponentially. By Corollary 3.2, $\lambda_*(G) > 1$. □

For Proposition 4.3 we need the following notations. Let ξ be a positive rational number such that $\xi \neq 1$ and denote by Q_ξ the smallest subgroup of the additive group of the rationals, which contains 1 and is invariant under multiplication by ξ and ξ^{-1}. In other words if $\xi = \frac{p}{q}$ with $p, q \in \mathbf{Z}$ and $\gcd(p, q) = 1$ then $Q_\xi \equiv \mathbf{Z}[\frac{1}{p}, \frac{1}{q}]$. Consider now the automorphism α of Q_ξ defined by $\alpha(x) = \xi x$, $x \in Q_\xi$. Let $\mathbf{Z}$ act on Q_ξ by powers of α. Denote by $G_\xi = Q_\xi \rtimes_\alpha \mathbf{Z}$ the corresponding semidirect product. The group G_ξ is a two-generated group with system of generators $\{\bar{a}, \bar{b}\}$, where $\bar{a} = 1 \in Q_\xi$ and the element $\bar{b}$ implements the automorphism α: $\bar{b}^{-1} x \bar{b} = \alpha(x)$, $x \in Q_\xi$.

Let now d be a natural number ≥ 2 and set $B_d = \prod_{\mathbf{Z}} \mathbf{Z}_d$. The group $\mathbf{Z}$ acts on B_d by shifts. The corresponding semidirect product $\Gamma(d)$, also denoted by $\mathbf{Z}_d \wr \mathbf{Z}$, is called the *wreath product* of $\mathbf{Z}$ and $\mathbf{Z}_d$. We shall consider $\Gamma(d)$ as generated by $\bar{a} = (\ldots, 0, 0, 1, 0, 0, \ldots)$ where 1 denotes a generator of $\mathbf{Z}_d$ (in the expression of $\bar{a}$ it appears at the 0-th coordinate place), and by $\bar{b}$, the element which implements the shift.

We have short exact sequences

$$0 \longrightarrow Q_\xi \longrightarrow G_\xi \longrightarrow \mathbf{Z} \longrightarrow 0$$
$$0 \longrightarrow B_d \longrightarrow \Gamma(d) \longrightarrow \mathbf{Z} \longrightarrow 0$$

so that G_ξ and $\Gamma(d)$ are two-step solvable. Slightly modifying the proof of Proposition 2.6 one gets

4.2. LEMMA. *The groups G_ξ and $\Gamma(d)$ have uniformly exponential growth.*

Our last class of two-generated one-relator groups of uniformly exponential growth is determined in the following statement.

4.3. PROPOSITION. *Let $G = \langle a, b; a^k [a, b]^l w(a, b) = 1 \rangle$ with $k, l \in \mathbf{Z}$ and $w(a, b) \in F^{(2)}$ where $F^{(2)} = [[F, F], [F, F]]$ denotes the second commutator subgroup of the free group $F = F(a, b)$ on a and b. Suppose that $(k, l) \notin \{\pm(2, 1), \pm(1, 1), \pm(1, 0), \pm(0, 1)\}$. Then G has uniformly exponential growth.*

Proof. Set $G_{k,l} = \langle a, b; a^k [a, b]^l w(a, b) \rangle$. Set also

$$\overline{G}_{k,l} = \langle a, b; a^k [a, b]^l w(a, b), F^{(2)} \rangle = \langle a, b; a^k [a, b]^l, F^{(2)} \rangle$$

which is a 2-step solvable quotient group of $G_{k,l}$. We shall show that $\overline{G}_{k,l}$ can be mapped homomorphically onto either G_ξ or $\Gamma(d)$ for a suitable positive rational number $\xi \neq 1$ or natural number $d \geq 2$.

Suppose first that $k \neq l, 2l$ and $lk \neq 0$. These assumptions guarantee that $\xi := \left|\frac{l-k}{l}\right| \neq 0, 1$. Then the map $a \longmapsto (\bar{a})^{sgn(\frac{l-k}{l})}, b \longmapsto \bar{b}$ from F onto G_ξ factorizes through $\overline{G}_{k,l}$. Indeed if we suppose, for instance, that $\frac{l-k}{l} > 0$, then the image of $a^k[a,b]^l$ is the number $k + l(-1+\xi) \in \mathbf{Q}_\xi$ which is zero. Thus $\overline{G}_{k,l}$ maps onto G_ξ.

Suppose now that $\gcd(k,l) = d$ or $(k,l) \in \{\pm(d,0), \pm(0,d)\}$ for some $d \geq 2$. Then, the same arguments as before show that $\overline{G}_{k,l}$ can be mapped onto $\Gamma(d)$ via the map $a \longmapsto \bar{a}, b \longmapsto \bar{b}$.

Finally observe that $\overline{G}_{0,0}$ is the free two-generated two-step solvable group $F/F^{(2)}$ and thus maps homomorphically onto $\Gamma(d)$ for any $d \geq 2$.

The proof follows from Lemma 4.2. □

Remark that the two-generated one-relator groups that are not covered by our statements have their relator that can be reduced to one of the form bw, $[a,b]w$ or $ba^{-1}baw$, where $w = w(a,b) \in F^{(2)}$.

Let us finish the paper by the following observation.

In [GrLP] it was conjectured that if G is a group with m generators and p relations, then

$$\lambda_*(G) \geq 2(m-p) - 1 .$$

For one-relator groups there is one case when Gromov's conjecture holds true.

4.4. PROPOSITION. *Let* $G = \langle a_1, a_2, \ldots, a_m : R(a_1, a_2, \ldots, a_m) = 1\rangle$, *with* $m \geq 2$, *be a one-relator group such that the relator* R *does not belong to the commutator subgroup* F' *of the free group* F *of rank* m *freely generated by* $a_1, a_2, \ldots, a_m$. *Then* $\lambda_*(G) \geq 2m - 3$.

Proof. We may assume that G is torsion-free. Indeed if $U, V \in F$ are such $U = V^k$ for some $k \in \mathbf{Z}$, then $U \in F'$ iff $V \in F'$. If the relator R is a proper power, say $R = W^k$, then G maps onto $G_1 = \langle a_1, a_2, \ldots, a_m : W(a_1, a_2, \ldots, a_m) = 1\rangle$, which is torsion-free, and $\lambda_*(G) \geq \lambda_*(G_1)$.

Under our assumptions on R, $H_1(G, \mathbf{Q}) \cong \mathbf{Z}^{m-1}$ and the second rational homology group $H_2(G, \mathbf{Q})$ vanishes.

In [S] it is proven that if $H_2(G, \mathbf{k}) = 0$, where $\mathbf{k}$ is a field, then any subset $\{x_j\} \in G$, whose image in $H_1(G, \mathbf{k})$ is linearly independent, freely generates a free group.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite system of generators for G. Then $\overline{X} = \{\bar{x}_1, \ldots, \bar{x}_n\}$, where $\bar{x}_i$ denotes the image of x_i in $H_1(G, \mathbf{Q})$, generates

$H_1(G,\mathbf{Q})$. We can find an independent subsystem $\{\bar{x}_{i_1},\dots\bar{x}_{i_{m-1}}\}$ in $H_1(G,\mathbf{Q})$ such that its pre-image $\{x_{i_1},\dots,x_{i_m}\}$ freely generates a free group. Therefore $\lambda_X(G)\geq 2(m-1)-1=2m-3$. $\square$

It seems to us that for a one-relator group G of rank $m\geq 3$ the inequality $\lambda_*(G)\geq 2m-3$ cannot be deduced directly from Magnus' Theorem as it is claimed in [GrLP].

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