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EXAMPLE 4. Similarly, $K = \mathbf{Q}\left(e\left(\frac{1}{8}\right), \sqrt{\pm q}\right)$ may be used to obtain strongly modular lattices $\Lambda$ in $G_{8r}(2^+q^+)$ from rank $r$ unimodular hermitian lattices over $\mathcal{O}_K$, setting now

$$x \cdot y = \frac{1}{2}\, Tr_{K/\mathbf{Q}}\big(h(x,y)/(2-\sqrt{2})\big)\,.$$

Again, we always have $\min \Lambda \geq 4$. E.g., $\mathcal{O}_K$ itself gives the tensor product of $D_4$ and the binary lattice with Gram matrix $\begin{pmatrix} 2 & 1 \\ 1 & (q+1)/2 \end{pmatrix}$.

## 2. ATKIN-LEHNER ACTION ON THETA FUNCTIONS

The subject treated in this section is not new, but appears to be difficult to cite from the literature (in the form we need it). For convenience, I give a rather detailed account, starting from a classical formula (due to Jacobi and others). Let $\Lambda$ be an even lattice. The theta function of a coset $\overline{v} = v + \Lambda$ in $\Lambda^*$ (and, in particular, $\Theta_\Lambda$ for $v = 0$) is that function defined on the upper half-plane by

$$\Theta_{\overline{v}}(z) = \sum_{x \in \overline{v}} e\left(\frac{1}{2}(x \cdot x)\, z\right)\,.$$

Now let $n = 2k$ ($k$ integral), and recall that $SL_2(\mathbf{R})$ acts on functions $f$ on the upper half-plane by

$$(f\,|_k\, S)(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right), \quad S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $S$ be in $SL_2(\mathbf{Z})$, with $c > 0$. For $u, v \in \Lambda^*$ define

$$\phi_S(u,v) = \sum_x e\big((ax \cdot x + 2x \cdot v + dv \cdot v)/2c\big)$$

where $x$ runs through a system of representatives of those elements of $\Lambda^*/c\Lambda$ which reduce to $\overline{u}$ in $D = \Lambda^*/\Lambda$. Each summand clearly depends only on the class $x + c\Lambda$, and the whole sum depends only on $\overline{u}$ and $\overline{v}$. The latter statement is trivial for $\overline{u}$, while for $\overline{v}$ it is proved (using $1 = ad - bc$) by

$$\begin{aligned} \phi_S(u,v) &= \sum_x e\big(a(x+dv)\cdot(x+dv)/2c\big)\, e\big(-b(2x\cdot v + dv \cdot v)/2\big) \\ &= \phi_S(u+dv, 0)\, e\big(-b(2u \cdot v + dv \cdot v)/2\big)\,. \end{aligned} \tag{2.1}$$

So we may write $\phi_S(u,v) = \phi_S(\overline{u}, \overline{v})$. Then the formula we need is (see [Mi], p. 189)

(2.2) $$\Theta_{\overline{u}}\big|_k S = (\det\Lambda)^{-\frac{1}{2}}(ic)^{-k}\sum_{\overline{v}\in D}\phi_S(\overline{u},\overline{v})\Theta_{\overline{v}}\,.$$

Let $\Lambda$ be of level $\ell$. Then it is a well-known consequence of (2.2) that $\Theta_\Lambda$ is a modular form for the subgroup $\Gamma_0(\ell)$ of $SL_2(\mathbf{Z})$ defined by $c \equiv 0 \pmod{\ell}$. Now let $m\|\ell$, $m' = \ell/m$. After [AL] the $m$-th Atkin-Lehner involution $\Gamma_0(\ell)W_m$, a coset of $\Gamma_0(\ell)$ in its normalizer in $SL_2(\mathbf{R})$, is given by any matrix of the form

$$W_m = S\begin{pmatrix}\sqrt{m} & 0\\ 0 & 1/\sqrt{m}\end{pmatrix}, \quad S = \begin{pmatrix}a\,b\\ c\,d\end{pmatrix}\in\Gamma_0(m'), \quad d \equiv 0 \,(\mathrm{mod}\ m)\,.$$

We specifically choose $a = 1, c = m'$, solve $tm + t'm' = 1$, and put $b = -t', d = tm$. In any case,

$$W_m^2 \equiv 1, \quad W_m W_{m'} \equiv W_\ell \,\big(\mathrm{mod}\ \Gamma_0(\ell)\big)\,.$$

Now these involutions are connected, via Gaussian sums, to those in Section 1 by the following important relation which seems to have been proved first by Kitaoka ([Ki] where, however, the constant factor is not worked out; cf. also [BS], p. 77, for the statement of a generalization).

ATKIN-LEHNER IDENTITY:

(2.3) $$\Theta_\Lambda\big|_k W_m = \left(\frac{\det\Lambda_m}{\det\Lambda}\right)^{\frac{1}{4}} i^{-k} g_{m'}(\Lambda_{m'})\Theta_{\Lambda_m}\,.$$

How this identity will be used here is readily explained. Suppose that $\Lambda$ (and then also $\Lambda_m$) has determinant $\ell^k$. We will be interested in the following gradually stronger conditions

(1) $\Theta_\Lambda = \Theta_{\Lambda_m}$ *for all* $m\|\ell$

(2) $\Lambda$ *is strongly modular.*

So condition (1) says that $\Theta_\Lambda$ is an eigenform of all the Atkin-Lehner involutions; Section 3 deals with such modular forms.

*Proof of* (2.3). Recall that here $S$ is chosen such that $a = 1$, $c = m'$ and $m|d$. Write $\overline{v}\in D$ in the form $\overline{v} = \overline{w} + \overline{y}$ where $\overline{w}\in D(m)$, $\overline{y}\in D(m')$. Then $\phi_S(0,\overline{v}) = \phi_S(0,\overline{y})$ by (2.1), and (2.2) gives

$$(\det\Lambda)^{\frac{1}{2}}(im')^k\Theta_\Lambda\big|_k S = \sum_{\overline{v}\in D}\phi_S(0,\overline{v})\Theta_{\overline{v}}$$
$$= \sum_{\overline{y}\in D(m')}\phi_S(0,\overline{y})\sum_{\overline{w}\in D(m)}\Theta_{\overline{w}+\overline{y}}\,.$$

We first must know that $\phi_S(0,\bar{y}) = 0$ for $\bar{y} \neq 0$. To see this, write $x \in \Lambda$ in the form $x = x_1 + m'x'$ with $x_1$ from some system of representatives of $\Lambda/\sqrt{m'}\Lambda_{m'}$ and $\sqrt{m'}x' \in \Lambda_{m'}$. Then

$$\phi_S(0,y)\, e(-dy\cdot y/2m') = \sum_{x_1} e\big((x_1\cdot x_1 + 2x_1\cdot y)/2m'\big) \sum_{\overline{x'}\in D(m')} e(x'\cdot y)\,.$$

But the sum over $D(m')$ vanishes for $\bar{y} \neq 0$ (since $D(m')$ is regular with respect to the discriminant form). It remains to determine $\phi_S(0,0)$. Put $E = \Lambda_{m'}^*/\Lambda_{m'}$. Since $(1/\sqrt{m'})\Lambda/\Lambda_{m'}$ is $E(m')$, the last formula for $y = 0$ gives

$$\begin{aligned}\phi_S(0,0) &= \#D(m') \sum_{\overline{v}\in E(m')} e\left(\frac{1}{2}v\cdot v\right)\\ &= \#D(m')\big(\#E(m')\big)^{\frac{1}{2}} g_{m'}(\Lambda_{m'})\end{aligned}$$

where

$$\begin{aligned}\#D(m')\big(\#E(m')\big)^{\frac{1}{2}} &= \big(\Lambda_{m'} : \sqrt{m'}\Lambda\big)\big(\sqrt{m'}\Lambda : m'\Lambda_{m'}\big)^{\frac{1}{2}}\\ &= \big(\#D(m')\big)^{\frac{1}{2}}(m')^k\\ &= \big(\#D(m)\big)^{-\frac{1}{2}}\big(\det\Lambda\big)^{\frac{1}{2}}(m')^k\\ &= m^{-\frac{k}{2}}\left(\frac{\det\Lambda_m}{\det\Lambda}\right)^{\frac{1}{4}}\big(\det\Lambda\big)^{\frac{1}{2}}(m')^k \qquad \square\end{aligned}$$

Note that, by Propositions 1 and 2, (2.3) in the case $\det\Lambda = \ell^k$ becomes

(2.4) $$\Theta_\Lambda\big|_k W_m = g_m(\Lambda_{m'})^{-1}\Theta_{\Lambda_m}\,.$$

## 3. Some use of modular forms

Let $W(\ell)$ be the elementary abelian 2-group formed by the Atkin-Lehner involutions $w_m = \Gamma_0(\ell)W_m$ for all $m\|\ell$. Let $k$ be even, and let $\mathcal{S}_k(\ell)$ denote the space of cusp forms of weight $k$ on $\Gamma_0(\ell)$. Then $W(\ell)$ acts on this space. For a character $\chi$ of $W(\ell)$ we let $\mathcal{S}_k(\ell)_\chi$ denote the subspace on which $W(\ell)$ acts by $\chi$. If $\Lambda$ and $M$ are lattices of dimension $2k$ and level $\ell$ belonging to the same genus, then $f = \Theta_\Lambda - \Theta_M$ is known to be in $\mathcal{S}_k(\ell)$, and when both lattices are strongly modular identity (2.4) implies that $f$ is in $\mathcal{S}_k(\ell)_\chi$ for the character $\chi(w_m) = g_m(\Lambda)$. So we are interested in such spaces now.

Fortunately, the dimension of $\mathcal{S}_k(\ell)_\chi$ is known; I am indebted to N.-P. Skoruppa for pointing out the reference. Let $s(\ell)$ denote the number of prime