

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 43 (1997)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** THE NORMALISER ACTION AND STRONGLY MODULAR LATTICES  
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**Kapitel:** 1. Introduction  
**DOI:** <https://doi.org/10.5169/seals-63272>

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## THE NORMALISER ACTION AND STRONGLY MODULAR LATTICES

by Gabriele NEBE<sup>\*</sup>)

ABSTRACT. We derive group theoretical methods to test if a lattice is strongly modular. We then apply these methods to the lattices of rational irreducible maximal finite groups.

### 1. INTRODUCTION

Let  $L \subseteq \mathbf{R}^d$  be an even integral lattice in the Euclidean space of dimension  $d$  and let  $L^\# \subseteq \mathbf{R}^d$  be its dual lattice. Let  $\pi(L)$  be the set of all intermediate lattices  $L \leq L' \leq L^\#$  that are inverse images of sums of Sylow subgroups of the finite abelian group  $L^\#/L$ . Then,  $L$  is said to be *strongly modular* if  $L$  is similar to  $L'$  for all  $L' \in \pi(L)$  (cf. [Que 96]). Recall that  $L$  and  $L'$  are called *similar* if there exists  $s \in GL(\mathbf{R}L)$  and  $a \in \mathbf{R}_{>0}$  such that  $Ls = L'$  and  $(vs, ws) = a(v, w)$  for all  $v, w \in \mathbf{R}L$ , where  $(,)$  denotes the Euclidean scalar product.

The automorphism group

$$G := \text{Aut}(L) = \{g \in O(\mathbf{R}L) \mid Lg \subseteq L\}$$

is conjugate to a finite subgroup of  $GL_d(\mathbf{Z})$ . Since  $G$  acts as group automorphisms on  $L^\#/L$  it preserves the lattices  $L' \in \pi(L)$ .

In Section 3 it is shown that the similarities  $L' \rightarrow L$  normalise  $G$ . So one may use the normaliser

$$N_{GL_d(\mathbf{Q})}(G) := \{n \in GL_d(\mathbf{Q}) \mid n^{-1}gn \in G \text{ for all } g \in G\}$$

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<sup>\*</sup>) Supported by the DFG.

of  $G$  in  $GL_d(\mathbf{Q})$  to test strong modularity of  $L$ . In the next section we derive some methods for explicitly constructing elements of  $N_{GL_d(\mathbf{Q})}(G)$ .

Every finite subgroup of  $GL_d(\mathbf{Q})$  is a subgroup of the automorphism group of an integral lattice. In particular the maximal finite subgroups of  $GL_d(\mathbf{Q})$  are automorphism groups of distinguished lattices. A subgroup of  $GL_d(\mathbf{Q})$  is called rational irreducible if it does not preserve a proper subspace  $\neq \{0\}$  of  $\mathbf{Q}^d$ . The rational irreducible maximal finite, abbreviated to *r.i.m.f.*, subgroups of  $GL_d(\mathbf{Q})$  are classified for  $d < 32$  (cf. [PIN 95], [NeP 95], [Neb 95], [Neb 96], [Neb 96a]). Their invariant lattices provide many examples of strongly modular lattices. The following theorem is proved by applying the methods derived in Section 4.

**THEOREM.** *In dimension  $d < 32$ , all even lattices  $L \subseteq \mathbf{R}^d$  that are preserved by a r.i.m.f. group and satisfy  $L^\# / L \cong (\mathbf{Z} / l\mathbf{Z})^{d/2}$  for some  $l \in \mathbf{N}$  are strongly modular, except for the lattices of the r.i.m.f. group  $[\pm \text{Alt}_6 . 2^2]_{16}$  in  $GL_{16}(\mathbf{Q})$  (cf. [NeP 95]).*

## 2. PRELIMINARIES AND NOTATION

The main strategy in this paper is the application of the following *normaliser principle*.

Let  $G$  be a group acting on a set  $S$ ,  $H$  a subgroup of the group of transformations of  $S$ . Then the normaliser of  $G$  in  $H$  acts on the set of  $G$ -orbits.

In our situation  $G = \text{Aut}(L)$  is the automorphism group of an integral lattice  $L$  in the Euclidean space  $\mathbf{R}L \cong \mathbf{R}^d$ . By writing the action of  $G$  on  $\mathbf{R}L$  with respect to a  $\mathbf{Z}$ -basis  $(b_1, \dots, b_d)$  of  $L$ ,  $G$  becomes a finite subgroup of  $GL_d(\mathbf{Z})$ . Then  $G = \text{Aut}(F) = \{g \in GL_d(\mathbf{Z}) \mid gFg^{tr} = F\}$  where  $F$  is the Gram matrix  $F = ((b_i, b_j))_{i,j=1}^d$  of  $L$ .

For the rest of this article let  $H = GL_d(\mathbf{Q})$ ,  $G \leq H$ , be a finite subgroup of  $H$ , and let  $N := N_H(G)$  be its normaliser. We also assume that  $G$  contains the negative unit matrix,  $-I_d \in G$ .

We apply the normaliser principle to the following three situations.

- (i)  $S = \{L \subseteq \mathbf{Q}^d \mid L = \sum_{i=1}^d \mathbf{Z}b_i \text{ for a basis } (b_1, \dots, b_d) \text{ of } \mathbf{Q}^d\}$ , the set of  $\mathbf{Z}$ -lattices of rank  $d$  in  $\mathbf{Q}^d$ , and the action of  $H$  on  $S$  is right multiplication:  $S \times H \rightarrow S$ ,  $(L, h) \mapsto Lh := \{lh \mid l \in L\}$ . Then the set of  $G$ -fixed points is

$$\mathcal{Z}(G) := \{L \in S \mid Lg = L \text{ for all } g \in G\},$$

the set of  $G$ -invariant lattices.