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For  $l \in L, v \in \mathbf{Q}^d$  we have  $vF(ln^{-1})^{tr} = va^{-1}nFl^{tr}$  and hence  $(L')^\# = L^\#an^{-1}$ .

Since  $L' \in \pi(L)$  one has  $L' = L^\# \cap a^{-1}L$ . Using this one obtains

$$Lan^{-2} = L'an^{-1} = L^\#an^{-1} \cap Ln^{-1} = (L')^\# \cap L' = L,$$

since  $(L')^\#/L$  is the orthogonal complement of  $L'/L$  in  $L^\#/L$  with respect to the induced quadratic form with values in  $\mathbf{Q}/\mathbf{Z}$ . So  $a^{-1}n^2 \in G$ .

Finally we check that  $n \in N$ . Let  $g \in G$ , then  $n^{-1}gn$  is in  $G = \text{Aut}(F)$  since  $Ln^{-1}gn = L'gn = L'n = L$  and

$$n^{-1}gnFn^{tr}g^{tr}n^{-tr} = n^{-1}agFg^{tr}n^{-tr} = F. \quad \square$$

#### 4. OBTAINING ELEMENTS OF $N$

Now we give examples as to how one may construct elements  $n$  of the normaliser  $N$ . To obtain similarities we are interested in  $n \in N$  of determinant  $\pm p^{d/2}$  for some (squarefree) natural number  $p$  such that  $p^{-1}n^2 \in G$ . The first method is an application of the normaliser principle to the situation (iii) described in Section 2:

**PROPOSITION 4.** *Let  $U \trianglelefteq G$  be a normal subgroup of  $G$  and assume that the commuting algebra  $K := C_{M_d(\mathbf{Q})}(U)$  is isomorphic to a number field. If  $c \in K$  satisfies  $c^2 = p \in \mathbf{Q}^*I_d$ , then  $c$  lies in  $N$ .*

*Proof.* Since  $G$  normalises  $U$ , it acts by conjugation (and hence as Galois automorphisms) on the abelian number field  $K$ . Now let  $c \in K$ , with  $c^2 =: p \in \mathbf{Q}^*I_d$  and  $g \in G$ . Then  $g$  stabilises the subfield  $\mathbf{Q}[c]$  and hence  $g^{-1}cg = \pm c$ , which is equivalent to  $c^{-1}gc = \pm g \in G$ . Therefore  $c \in N$ , since we assumed that  $-I_d \in G$ .  $\square$

The following construction described in [PIN 95] Proposition (II.4) also allows us to find elements of  $N$ .

For  $i = 1, 2$  let  $G_i \leq GL_{d_i}(\mathbf{Q})$  be finite rational irreducible matrix groups with commuting algebras  $A_i \subseteq M_{d_i}(\mathbf{Q})$ . Also let  $Q$  be a maximal common subalgebra of dimension  $z$  of  $A_1$  and  $A_2$ . Let  $d := \frac{d_1d_2}{z}$  and view the  $G_i$  as subgroups of  $G_1 \otimes_Q G_2 \leq GL_d(\mathbf{Q})$ . If there exist elements  $a_i \in N_{GL_d(\mathbf{Q})}(G_i)$

centralising  $G_j$  and  $a_j$  ( $1 \leq i \neq j \leq 2$ ) and a squarefree natural number  $p \neq 0$  such that  $p^{-1}a_i^2 \in G_i$ , the group

$$G := \langle G_1 \otimes_{\mathbf{Q}} G_2, p^{-1}a_1a_2 \rangle,$$

generated by the elements of  $G_1 \otimes_{\mathbf{Q}} G_2$  and  $p^{-1}a_1a_2$ , is a finite subgroup of  $GL_d(\mathbf{Q})$  containing  $G_1 \otimes_{\mathbf{Q}} G_2$  as a subgroup of index 2.

For  $d \leq 31$  and  $p > 1$  we only need the case where  $a_2$  is an element of the enveloping algebra of  $G_2$ . Then  $G$  is denoted by  $G_1 \otimes_{\mathbf{Q}}^{2(p)} G_2$  (or  $G_1 \boxtimes_{\mathbf{Q}}^{2(p)} G_2$ ) according to whether  $a_1$  is (or is not) a rational linear combination of elements of  $G_1$ .

Using this notation one immediately has the following proposition.

**PROPOSITION 5.** *For  $i = 1, 2$  the matrix  $a_i$  is an element of determinant  $\pm p^{d/2}$  in the normaliser  $N$  of  $G$ .*

A common feature of the situations in Propositions 4 and 5 is that we extend the natural representation of  $G$  to a projective representation which is realisable as a linear representation over a quadratic extension of  $\mathbf{Q}$ .

**PROPOSITION 6.** *Let  $G \trianglelefteq E$  be a supergroup containing  $G$  of index 2. Assume that  $C_{M_d(\mathbf{Q})}(G) \cong \mathbf{Q}$  and that the natural character of  $G$  extends to  $E$  with character field  $\mathbf{Q}[\sqrt{p}]$ , where  $p \in \mathbf{Z}$  is not a square. Then there exists  $n \in N$  of determinant  $\pm p^{d/2}$  with  $p^{-1}n^2 \in G$ .*

*Proof.* By Clifford theory one may extend the natural representation  $\Delta$  of  $G$  to a representation  $\delta_1 \otimes \delta_2 : E \rightarrow (\mathbf{Q}[\sqrt{p}] \otimes M_d(\mathbf{Q}))^*$ , where  $\delta_1$  and  $\delta_2$  are projective representations  $\delta_1(G) = \{1\}$  and  $(\delta_2)|_G = \Delta$ . Let  $e \in E \setminus G$ . Then

$$(\delta_1(e) \otimes \delta_2(e))^2 = \delta_1(e)^2 \otimes \delta_2(e)^2 = 1 \otimes \Delta(e^2),$$

since  $e^2 \in G$ . Therefore  $\delta_1(e)^2 \in \mathbf{Q}$ . Replacing  $\delta_1(e)$  by a suitable rational multiple (and multiplying  $\delta_2(e)$  by the inverse) one may assume that  $\delta_1(e)^2 = p^{-1}$ . Then  $n := \delta_2(e)$  is an element of the normaliser  $N$  with the desired properties.  $\square$

	Aut(L)	det(L)	min(L)	$ L_{\min} $	lattice sparse
4	$F_4 \tilde{\otimes} F_4 = [2_+^{1+8} \cdot O_8^+(2)]_{16}$	$2^8$	4	4320	+
6	$[(SL_2(9) \otimes_{\infty,3}^{2(3)} SL_2(9)) \cdot 2]_{16}$	$3^8$	4	720	+
9	$[(Sp_4(3) \circ C_3) \boxtimes_{\sqrt{-3}}^2 SL_2(3)]_{16}$	$2^8 \cdot 3^8$	6	960	+
14	$[2 \cdot \text{Alt}_{10}]_{16}$	$5^8$	6	2400	+
16	$[SL_2(5) \otimes_{\infty,2}^{2(2)} 2^{1+4'} \cdot \text{Alt}_5]_{16}$	$2^8 \cdot 5^8$	8	1200	+
19	$[SL_2(5) \otimes_{\infty,3}^{2(3)} SL_2(9)]_{16}$	$3^8 \cdot 5^8$	10	1440	+
21	$[SL_2(5) \otimes_{\infty,3}^{2(3)} (SL_2(3) \square^2 C_3)]_{16}$	$2^8 \cdot 3^8 \cdot 5^8$	12	480	+
25	$[2 \cdot \text{Alt}_7 \otimes_{\sqrt{-7}}^{2(3)} \tilde{S}_3]_{16}$	$3^8 \cdot 7^8$	12	1680	+
26	$[SL_2(7) \otimes_{\sqrt{-7}}^{2(3)} \tilde{S}_3]_{16}$	$3^8 \cdot 7^8$	10	336	$p \neq 2$
3	$[2 \cdot \text{Co}_1]_{24}$	$1^{24}$	4	196560	+
6	$[6 \cdot U_4(3) \cdot 2 \boxtimes_{\sqrt{-3}}^2 SL_2(3)]_{24}$	$2^{12}$	4	3024	$p \neq 3$
16	$[6 \cdot L_3(4) \cdot 2 \otimes^{2(2)} D_8]_{24}$	$2^{12} \cdot 3^{12}$	8	3024 + 7560	+
17	$[(SL_2(3) \circ C_4) \cdot 2 \boxtimes_{\sqrt{-1}}^{2(3)} U_3(3)]_{24}$	$2^{12} \cdot 3^{12}$	8	4536 + 6048	+
18	$A_{24}$	$1^{24}$	2	600	$p \neq 5$
22	$[2 \cdot J_2 \square^2 SL_2(5)]_{24}$	$5^{12}$	8	37800	+
35	$[L_2(7) \otimes^{2(2)} F_4]_{24}$	$7^{12}$	8	1008 + 3024	$p \neq 2$
40	$[SL_2(13) \square^{2(2)} SL_2(3)]_{24}$	$13^{12}$	12	$2 \cdot 2184 + 8736$	$p \neq 2$
42	$[6 \cdot \text{Alt}_7 : 2]_{24}$	$2^{12}$	4	3024	+
43	$[3 \cdot M_{10} \otimes_{\sqrt{-3}}^{2(2)} SL_2(3)]_{24}$	$2^{12} \cdot 5^{12}$	8	1080	$p \neq 3$
44	$[\text{Alt}_5 \otimes_{\sqrt{5}}^2 (C_3 \otimes^{2(2)} D_8)]_{24}$	$2^{12} \cdot 3^{12} \cdot 5^{12}$	16	$360 + 2 \cdot 720$	$p \neq 2$
45	$[3 \cdot M_{10} \otimes^{2(2)} D_8]_{24}$	$2^{12} \cdot 3^{12} \cdot 5^{12}$	16	$1080 + 1080$	+
64	$[SL_2(11) \otimes_{\sqrt{-11}}^{2(2)} SL_2(3)]_{24}$	$2^{12} \cdot 11^{12}$	12	1320	$p \neq 2$