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For $l \in L, v \in \mathbf{Q}^d$ we have $vF(ln^{-1})^{tr} = va^{-1}nFl^{tr}$ and hence $(L')^{\#} = L^{\#}an^{-1}$.

Since $L' \in \pi(L)$ one has $L' = L^{\#} \cap a^{-1}L$. Using this one obtains

$$Lan^{-2} = L'an^{-1} = L^{\#}an^{-1} \cap Ln^{-1} = (L')^{\#} \cap L' = L,$$

since $(L')^{\#}/L$ is the orthogonal complement of L'/L in $L^{\#}/L$ with respect to the induced quadratic form with values in $\mathbf{Q}/\mathbf{Z}$. So $a^{-1}n^2 \in G$.

Finally we check that $n \in N$. Let $g \in G$, then $n^{-1}gn$ is in $G = \mathrm{Aut}(F)$ since $Ln^{-1}gn = L'gn = L'n = L$ and

$$n^{-1}gnFn^{tr}g^{tr}n^{-tr} = n^{-1}agFg^{tr}n^{-tr} = F. \qquad \square$$

4. Obtaining Elements of N

Now we give examples as to how one may construct elements n of the normaliser N. To obtain similarities we are interested in $n \in N$ of determinant $\pm p^{d/2}$ for some (squarefree) natural number p such that $p^{-1}n^2 \in G$. The first method is an application of the normaliser principle to the situation (iii) described in Section 2:

PROPOSITION 4. *Let $U \trianglelefteq G$ be a normal subgroup of G and assume that the commuting algebra $K := C_{M_d(\mathbf{Q})}(U)$ is isomorphic to a number field. If $c \in K$ satisfies $c^2 = p \in \mathbf{Q}^* I_d$, then c lies in N.*

Proof. Since G normalises U, it acts by conjugation (and hence as Galois automorphisms) on the abelian number field K. Now let $c \in K$, with $c^2 =: p \in \mathbf{Q}^* I_d$ and $g \in G$. Then g stabilises the subfield $\mathbf{Q}[c]$ and hence $g^{-1}cg = \pm c$, which is equivalent to $c^{-1}gc = \pm g \in G$. Therefore $c \in N$, since we assumed that $-I_d \in G$. $\square$

The following construction described in [PlN 95] Proposition (II.4) also allows us to find elements of N.

For $i = 1, 2$ let $G_i \leq GL_{d_i}(\mathbf{Q})$ be finite rational irreducible matrix groups with commuting algebras $A_i \subseteq M_{d_i}(\mathbf{Q})$. Also let Q be a maximal common subalgebra of dimension z of A_1 and A_2. Let $d := \frac{d_1 d_2}{z}$ and view the G_i as subgroups of $G_1 \underset{Q}{\otimes} G_2 \leq GL_d(\mathbf{Q})$. If there exist elements $a_i \in N_{GL_d(\mathbf{Q})}(G_i)$

centralising G_j and a_j $(1 \leq i \neq j \leq 2)$ and a squarefree natural number $p \neq 0$ such that $p^{-1}a_i^2 \in G_i$, the group

$$G := \langle G_1 \underset{Q}{\otimes} G_2, p^{-1}a_1a_2 \rangle ,$$

generated by the elements of $G_1 \underset{Q}{\otimes} G_2$ and $p^{-1}a_1a_2$, is a finite subgroup of $GL_d(\mathbf{Q})$ containing $G_1 \underset{Q}{\otimes} G_2$ as a subgroup of index 2.

For $d \leq 31$ and $p > 1$ we only need the case where a_2 is an element of the enveloping algebra of G_2. Then G is denoted by $G_1 \overset{2(p)}{\underset{Q}{\otimes}} G_2$ (or $G_1 \overset{2(p)}{\underset{Q}{\boxtimes}} G_2$) according to whether a_1 is (or is not) a rational linear combination of elements of G_1.

Using this notation one immediately has the following proposition.

PROPOSITION 5. *For $i = 1, 2$ the matrix a_i is an element of determinant $\pm p^{d/2}$ in the normaliser N of G.*

A common feature of the situations in Propositions 4 and 5 is that we extend the natural representation of G to a projective representation which is realisable as a linear representation over a quadratic extension of $\mathbf{Q}$.

PROPOSITION 6. *Let $G \trianglelefteq E$ be a supergroup containing G of index 2. Assume that $C_{M_d(\mathbf{Q})}(G) \cong \mathbf{Q}$ and that the natural character of G extends to E with character field $\mathbf{Q}[\sqrt{p}]$, where $p \in \mathbf{Z}$ is not a square. Then there exists $n \in N$ of determinant $\pm p^{d/2}$ with $p^{-1}n^2 \in G$.*

Proof. By Clifford theory one may extend the natural representation Δ of G to a representation $\delta_1 \otimes \delta_2 : E \to \left(\mathbf{Q}[\sqrt{p}] \otimes M_d(\mathbf{Q})\right)^*$, where δ_1 and δ_2 are projective representations $\delta_1(G) = \{1\}$ and $(\delta_2)_{|G} = \Delta$. Let $e \in E \setminus G$. Then

$$\left(\delta_1(e) \otimes \delta_2(e)\right)^2 = \delta_1(e)^2 \otimes \delta_2(e)^2 = 1 \otimes \Delta(e^2),$$

since $e^2 \in G$. Therefore $\delta_1(e)^2 \in \mathbf{Q}$. Replacing $\delta_1(e)$ by a suitable rational multiple (and multiplying $\delta_2(e)$ by the inverse) one may assume that $\delta_1(e)^2 = p^{-1}$. Then $n := \delta_2(e)$ is an element of the normaliser N with the desired properties. $\square$

	Aut(L)	det(L)	min(L)	$\lvert L_{\min}\rvert$	lattice sparse
4	$F_4 \tilde{\otimes} F_4 = [2_+^{1+8}.O_8^+(2)]_{16}$	2^8	4	4320	$+$
6	$[(SL_2(9) \underset{\infty,3}{\overset{2(3)}{\otimes}} SL_2(9)).2]_{16}$	3^8	4	720	$+$
9	$[(Sp_4(3)\circ C_3) \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} SL_2(3)]_{16}$	$2^8\cdot 3^8$	6	960	$+$
14	$[2.\mathrm{Alt}_{10}]_{16}$	5^8	6	2400	$+$
16	$[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2^{1+4'}.\mathrm{Alt}_5]_{16}$	$2^8\cdot 5^8$	8	1200	$+$
19	$[SL_2(5) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} SL_2(9)]_{16}$	$3^8\cdot 5^8$	10	1440	$+$
21	$[SL_2(5) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} (SL_2(3) \overset{2}{\square} C_3)]_{16}$	$2^8\cdot 3^8\cdot 5^8$	12	480	$+$
25	$[2.\mathrm{Alt}_7 \underset{\sqrt{-7}}{\overset{2(3)}{\boxtimes}} \tilde{S}_3]_{16}$	$3^8\cdot 7^8$	12	1680	$+$
26	$[SL_2(7) \underset{\sqrt{-7}}{\overset{2(3)}{\boxtimes}} \tilde{S}_3]_{16}$	$3^8\cdot 7^8$	10	336	$p\neq 2$
3	$[2.Co_1]_{24}$	1^{24}	4	196560	$+$
6	$[6.U_4(3).2 \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} SL_2(3)]_{24}$	2^{12}	4	3024	$p\neq 3$
16	$[6.L_3(4).2 \overset{2(2)}{\otimes} D_8]_{24}$	$2^{12}\cdot 3^{12}$	8	$3024+7560$	$+$
17	$[(SL_2(3)\circ C_4).2 \underset{\sqrt{-1}}{\overset{2(3)}{\boxtimes}} U_3(3)]_{24}$	$2^{12}\cdot 3^{12}$	8	$4536+6048$	$+$
18	A_{24}	1^{24}	2	600	$p\neq 5$
22	$[2.J_2 \overset{2}{\square} SL_2(5)]_{24}$	5^{12}	8	37800	$+$
35	$[L_2(7) \overset{2(2)}{\boxtimes} F_4]_{24}$	7^{12}	8	$1008+3024$	$p\neq 2$
40	$[SL_2(13) \overset{2(2)}{\square} SL_2(3)]_{24}$	13^{12}	12	$2\cdot 2184+8736$	$p\neq 2$
42	$[6.\mathrm{Alt}_7 : 2]_{24}$	2^{12}	4	3024	$+$
43	$[3.M_{10} \underset{\sqrt{-3}}{\overset{2(2)}{\boxtimes}} SL_2(3)]_{24}$	$2^{12}\cdot 5^{12}$	8	1080	$p\neq 3$
44	$[\mathrm{Alt}_5 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{24}$	$2^{12}\cdot 3^{12}\cdot 5^{12}$	16	$360+2\cdot 720$	$p\neq 2$
45	$[3.M_{10} \overset{2(2)}{\boxtimes} D_8]_{24}$	$2^{12}\cdot 3^{12}\cdot 5^{12}$	16	$1080+1080$	$+$
64	$[SL_2(11) \underset{\sqrt{-11}}{\overset{2(2)}{\boxtimes}} SL_2(3)]_{24}$	$2^{12}\cdot 11^{12}$	12	1320	$p\neq 2$