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calculate the quadrilateral, pentagon and hexagon spaces. Section 7 lists some open problems.

The study of the polygon spaces will be pursued in a forthcoming paper [HK] in which we shall compute the cohomology ring of these spaces.

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2. THE POLYGON SPACES

(2.1) Let V be a real vector space and m a positive integer. Let ${}^m\mathcal{F}(V)$ be the real vector space of all maps $\rho\colon \{1,2,\dots,m\} \to V$ such that $\sum_{j=1}^m \rho(j) = 0$. An element $\rho \in {}^m\mathcal{F}(V)$ will be regarded as a closed polygonal path in V

$$0 \bullet\!\!\rightarrowtail\!\!\bullet\ \rho(1) \bullet\!\!\rightarrowtail\!\!\bullet\ \rho(1)+\rho(2) \bullet\!\!\rightarrowtail\!\!\bullet \cdots \bullet\!\!\rightarrowtail\!\!\bullet \sum_{j=1}^{m} \rho(j) = 0$$

of m steps, or, alternately, as a configuration in V (up to translation) of a polygon of m sides. We shall call an element $\rho \in {}^m\mathcal{F}(V)$ an *m-polygon (in V)* and a *proper polygon* when $\rho(j) \neq 0\ \forall j$. We use the notation ${}^m\mathcal{F}^k$ for the space ${}^m\mathcal{F}(\mathbf{R}^k)$.

The group $\mathbf{R}_+$ of positive homotheties of V acts freely and properly on ${}^m\mathcal{F}(V) - \{0\}$. The quotient ${}^m\widetilde{\mathcal{P}}(V) := ({}^m\mathcal{F}(V) - \{0\})/\mathbf{R}_+$ then inherits a structure of smooth manifold diffeomorphic to a sphere. For instance, ${}^m\widetilde{\mathcal{P}}^k := ({}^m\mathcal{F}^k - \{0\})/\mathbf{R}_+$ is diffeomorphic to the sphere $S^{k(m-1)-1}$.

(2.2) Suppose now that V is oriented and is a Euclidean space, namely V is endowed with a scalar product. The group $O(V)$ of isometries of V acts on ${}^k\mathcal{F}^m$ and ${}^m\widetilde{\mathcal{P}}(V)$; we define the moduli spaces

$${}^m\mathcal{P}(V)_+ := SO(V)\backslash{}^m\widetilde{\mathcal{P}}(V) \quad \text{and} \quad {}^m\mathcal{P}(V) := O(V)\backslash{}^m\widetilde{\mathcal{P}}(V)$$

of m-polygons in V, up to similitude (where $SO(V)$ is the identity component of $O(V)$). Observe that any orientation preserving isometry $h\colon V \xrightarrow{\sim} \mathbf{R}^k$ produces identifications

$$^m\mathcal{P}(V)_+ \simeq {}^m\mathcal{P}^k_+ := SO_k\backslash {}^m\widetilde{\mathcal{P}}^k \quad \text{and} \quad {}^m\mathcal{P}(V) \simeq {}^m\mathcal{P}^k := O_k\backslash {}^m\widetilde{\mathcal{P}}^k .$$

We shall use the fact that these identifications do not depend on the choice of h and thus ${}^m\mathcal{P}(V)_+$ and ${}^m\mathcal{P}(V)$, for any Euclidean space V, are canonically identified with ${}^m\mathcal{P}^k_+$ and ${}^m\mathcal{P}^k$.

(2.3) The "degree of improperness" of polygons provides a stratification

$$\varnothing = E_1{}^m\widetilde{\mathcal{P}}(V) \subset E_2{}^m\widetilde{\mathcal{P}}(V) \subset \cdots \subset E_{m-1}{}^m\widetilde{\mathcal{P}}(V) \subset E_m{}^m\widetilde{\mathcal{P}}(V) = {}^m\widetilde{\mathcal{P}}(V)$$

where

$$E_j{}^m\widetilde{\mathcal{P}}(V) := \{\rho \in {}^m\widetilde{\mathcal{P}}(V) \mid \sharp\{s \mid \rho(s) = 0\} \geq m - j\} .$$

The "open stratum" $E_j{}^m\widetilde{\mathcal{P}}(V) - E_{j-1}{}^m\widetilde{\mathcal{P}}(V)$ is a smooth submanifold of ${}^m\widetilde{\mathcal{P}}(V)$ of dimension $(j-1)k-1$ if $k = \dim V$. The top open stratum ${}^m\widetilde{\mathcal{P}}(V) - E_{m-1}{}^m\widetilde{\mathcal{P}}(V)$, open and dense in ${}^m\widetilde{\mathcal{P}}(V)$, is the space of proper polygons.

As this stratification is $O(V)$-invariant, it projects onto stratifications $\{E_j{}^m\mathcal{P}^k_+\}$ and $\{E_j{}^m\mathcal{P}^k\}$ of the moduli spaces (using the canonical identifications of (2.2)). We shall see in §3 that the above stratifications describe the singular loci of smooth orbifold structures on the spaces ${}^m\widetilde{\mathcal{P}}(V)$, ${}^m\mathcal{P}^k_+$ and ${}^m\mathcal{P}^k$.

(2.4) The map $\rho \mapsto |\rho| := \sum_{j=1}^m |\rho(j)|$ which associates to a polygon ρ its total perimeter is a norm on ${}^m\mathcal{F}(V)$. We denote by $S({}^m\mathcal{F}(V))$ the sphere of radius 2 for this norm. Each class in ${}^m\widetilde{\mathcal{P}}(V)$ has a unique representative in $S({}^m\mathcal{F}(V))$ which gives a topological embedding $\imath\colon {}^m\widetilde{\mathcal{P}}(V) \longrightarrow {}^m\mathcal{F}(V)$ whose image is $S({}^m\mathcal{F}(V))$. The image by $\imath$ of $E_{m-1}{}^m\widetilde{\mathcal{P}}(V)$ is the subset of $S({}^m\mathcal{F}(V))$ where $S({}^m\mathcal{F}(V))$ fails to be a smooth submanifold of ${}^m\mathcal{F}(V)$. However, the restriction of $\imath$ to each $E_j{}^m\widetilde{\mathcal{P}}(V) - E_{j-1}{}^m\widetilde{\mathcal{P}}(V)$ is a smooth embedding.

The map $\widetilde{\ell}\colon {}^m\mathcal{F}(V) \to \mathbf{R}^m$ defined by $\widetilde{\ell}(\rho) := (|\rho(1)|, \dots, |\rho(m)|)$ associates to a polygon its side-lengths. We define $\ell : {}^m\widetilde{\mathcal{P}}(V) \longrightarrow \mathbf{R}^m$ by $\ell := \widetilde{\ell} \circ \imath$. We shall also use the notation $\ell_i(\rho)$ for $|\rho(i)|$. These maps are invariant under the $O(V)$-action and thus define maps (always called ℓ)

$$\ell : {}^m\mathcal{P}^k_+ \longrightarrow \mathbf{R}^m \quad \text{and} \quad \ell : {}^m\mathcal{P}^k \longrightarrow \mathbf{R}^m$$

which are smooth on each open stratum.

(2.5) Let $\tau\colon V \longrightarrow V$ be the orthogonal reflection through some hyperplane Π in V. One has the involution $\rho \mapsto \check{\rho} := \tau \circ \rho$ on ${}^m\mathcal{F}(V)$ and ${}^m\widetilde{\mathcal{P}}(V)$ whose fixed-point space is naturally ${}^m\mathcal{F}(\Pi)$ and ${}^m\widetilde{\mathcal{P}}(\Pi)$. If $h \in SO(V)$, one has

$$\tau \circ h = \underbrace{(\tau \circ h \circ \tau \circ h^{-1})}_{\in SO(V)} \circ h \circ \tau .$$

Hence the involution descends to an involution (still denoted $\rho \mapsto \check{\rho}$) on ${}^m\mathcal{P}^k_+$. If τ' is an orthogonal reflection with respect to another hyperplane Π', then the formula $\tau \circ \rho' = (\tau' \circ \tau) \circ \tau \circ \rho$ shows that the induced involution on ${}^m\mathcal{P}^k_+$ does not depend on the choice of τ. The fixed point space of $\check{}$ is ${}^m\mathcal{P}^{k-1}$. Observe that $\check{\rho} = \rho$ in ${}^m\mathcal{P}^k$.

EXAMPLES

(2.6) *Polygons in the line.* The space ${}^m\widetilde{\mathcal{P}}^1 = {}^m\widetilde{\mathcal{P}}^1_+$ is diffeomorphic to the sphere S^{m-2}. Under this identification, the O_1-action becomes the antipodal map and thus ${}^m\mathcal{P}^1$ is a smooth manifold diffeomorphic to $\mathbf{R}P^{m-2}$. For example, ${}^3\widetilde{\mathcal{P}}^1 \simeq S^1$ and ${}^3\mathcal{P}^1 \simeq \mathbf{R}P^1$. The stratum $E_2{}^3\widetilde{\mathcal{P}}^1$ consists of 3 pairs of antipodal points and thus $E_2{}^3\mathcal{P}^1$ is a set of 3 points, the three triangles with one side of length 0. This corresponds to the fact that $S({}^3\mathcal{F}^1)$ is a regular hexagon and $O_1 \backslash S({}^3\mathcal{F}^1)$ is a triangle. Actually, the map $\ell : {}^3\mathcal{P}^1 \longrightarrow \mathbf{R}^3$ produces homeomorphisms

$${}^3\mathcal{P}^1 \underset{\simeq}{\xrightarrow{\ \ell\ }} \{(x,y,z) \in \mathbf{R}^3_{\geq 0} \mid x+y+z = 2 \quad \text{and} \quad \pm x \pm y \pm z = 0\} .$$

(2.7) *Polygons in the plane.* Identifying $\mathbf{R}^2$ with $\mathbf{C}$, the space ${}^m\mathcal{F}^2$ is a complex vector space isomorphic to $\mathbf{C}^{m-1}$ and the (free) SO_2-action corresponds to the diagonal U_1-action. As in (2.6) one establishes the diffeomorphisms

$$\begin{array}{ccc} {}^m\widetilde{\mathcal{P}}^2 & \xrightarrow{\ \simeq\ } & S^{2m-3} \\ \downarrow & & \downarrow \\ {}^m\mathcal{P}^2_+ & \xrightarrow{\ \simeq\ } & \mathbf{C}P^{m-2} \end{array}$$

The above diffeomorphisms conjugate the involutions $\check{}$ with the complex conjugations of $\mathbf{C}^{m-1}$ and $\mathbf{C}P^{m-2}$. Also, the involution $\check{}$ on ${}^m\mathcal{P}^2_+$ coincides with the residual O_1 action and therefore ${}^m\mathcal{P}^2$ is the quotient of $\mathbf{C}P^{m-2}$ by its complex conjugation.

For example, ${}^3\widetilde{\mathcal{P}}^2$, the space of planar triangles, is diffeomorphic to the sphere S^3. The singular stratum $E_2({}^3\widetilde{\mathcal{P}}^1)$ is a link of three circles which are SO_2-orbits (therefore, any two of them constitute a Hopf link). The quotient ${}^3\mathcal{P}^2_+$ is identified with $\mathbf{C}P^1$ and $E_2({}^3\widetilde{\mathcal{P}}^1_+)$ is a set of three points in $\mathbf{C}P^1$.

Finally, ${}^3\mathcal{P}^2 \simeq \mathbf{C}P^1/\{z \sim \bar{z}\}$ is homeomorphic, via the length-side map ℓ, to the solid triangle

$$ {}^3\mathcal{P}^2 = {}^3\mathcal{P}^3 \xrightarrow[\simeq]{\ell} \{(x_1,x_2,x_3) \in \mathbf{R}^3 \mid x_1+x_2+x_3 = 2 \quad \text{and} \quad 0 \le x_i \le 1\} $$

with boundary ${}^3\mathcal{P}^1$.

3. Quaternions, Grassmannians and structures on the full polygon spaces

(3.1) Let $\mathbf{H} = \mathbf{C} \oplus \mathbf{C}j$ be the skew-field of quaternions; the space $I\mathbf{H}$ of pure imaginary quaternions is equipped with the orthonormal basis i, j and $k = ij$, giving rise to an isometry with $\mathbf{R}^3$ which turns the pure imaginary part of the quaternionic multiplication pq into the usual cross product $p \times q$. The space ${}^m\mathcal{F}^3$ is thus identified with ${}^m\mathcal{F}(I\mathbf{H})$ which gives rise to the canonical identifications on the the various moduli spaces (see (2.2)).

Recall that the correspondence

$$ \eta : u + vj \mapsto \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} $$

gives an injective $\mathbf{R}$-algebra homomorphism $\eta : \mathbf{H} \longrightarrow \mathcal{M}_{(2\times 2)}(\mathbf{C})$. This enables a matrix $P \in U_2$ to act on the right or on the left on $\mathbf{H}$. It also identifies the group S^3 of unit quaternions with SU_2.

(3.2) The Hopf map $\phi : \mathbf{H} \longrightarrow I\mathbf{H}$ defined by

$$ \phi(q) := \bar{q}\, i\, q $$

sends the 3-sphere of radius $\sqrt{r}$ in $\mathbf{H}$ onto the 2-sphere of radius r in $I\mathbf{H}$. (The formulae given in the original paper by Hopf [Ho, § 5] actually correspond to the map $q \mapsto \bar{q}kq$.) The equality $\phi(q) = \phi(q')$ occurs if and only if $q' = e^{i\theta} q$. The map ϕ satisfies the equivariance relation $\phi(q \cdot P) = P^{-1} \cdot \phi(q) \cdot P$. Writing $q = u + vj$ with $u, v \in \mathbf{C}$, one has

$$ \phi(u+vj) = (\bar{u} - j\bar{v})\, i(u+vj) = i(\bar{u} + j\bar{v})(u+vj) = i\big[(|u|^2 - |v|^2) + 2\bar{u}vj\big] . $$

(3.3) Observe that if $q = s + tj$ with $s, t \in \mathbf{R}$, then $\phi(q) = i\,q^2$. This plane $\mathbf{R} \oplus \mathbf{R}j$ of its images is the fixed point set of the involution $a + bj \mapsto \bar{a} + \bar{b}j$ that will be used later. Its image under ϕ is $\mathbf{R}i \oplus \mathbf{R}k$.

(3.4) Remark. $I\mathbf{H}$, with the Lie bracket $[p,q] = pq - qp = 2\,\mathrm{Im}(pq)$, is the Lie algebra for the group $U_1(\mathbf{H}) \simeq SU_2 \simeq S^3$. The pairing