

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 44 (1998)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: DYNAMICAL SYSTEMS APPROACH TO BIRKHOFF'S THEOREM
Autor: SIBURG, Karl Friedrich
Kapitel: 3. CONCLUDING REMARKS
DOI: <https://doi.org/10.5169/seals-63906>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

Download PDF: 02.04.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

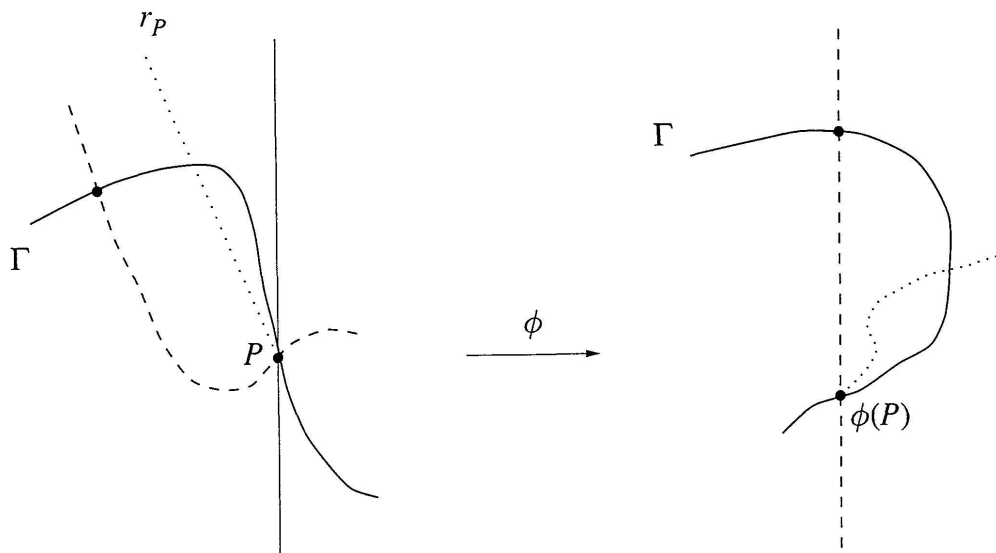


FIGURE 5

Why Γ must be a Lipschitz graph

III. THE SECOND CASE

Finally, the same remark can be applied in the second of the two cases from Lemma 1 where Γ contains a whole vertical interval. For we may take P to be the midpoint of that interval and apply ϕ once – the vertical through $\phi(P)$ will intersect Γ in two isolated points D_0 and E_0 , and we are back in the first situation we already dealt with.

The proof of the theorem is complete.

3. CONCLUDING REMARKS

For the sake of clarity, we did not prove the most general result that can be obtained by our method. Here we just indicate possible generalizations.

First of all, our proof does not require the monotone twist condition but only a sort of “cone condition on Γ ”. Namely, what we really need is the requirement that all (pre-)images of verticals lie outside certain cones centred at points on Γ ; we do not use the much more restrictive fact that they are graphs. (This subtle point might be the reason why we have not succeeded in proving a well-known generalization of Birkhoff’s Theorem to boundaries of invariant annuli [Fa, He, KH] by our method.)

Secondly, Birkhoff's Theorem also holds true for invariant curves of products $\phi_N \circ \dots \circ \phi_1$ of monotone twist mappings *of the same sign*. In general, such products are not monotone twist mappings anymore. This generalization follows immediately by our method if, even more generally, each ϕ_n satisfies the same "cone condition" on $(\phi_{n-1} \circ \dots \circ \phi_1)(\Gamma)$. For every single ϕ_n presses more area into a fold, and $\sup_{n \geq 0} |\Omega_n| < \infty$ because Γ is mapped onto itself again after N steps, instead of one. A proof along the traditional lines was given by Mather only a couple of years ago [Ma3, Appendix].

Finally, we did not really need that ϕ is a diffeomorphism. Everything can also be formulated and proved for homeomorphisms that preserve Lebesgue measure and satisfy the "cone condition".

REFERENCES

- [Bi1] BIRKHOFF, G. D. Surface transformations and their dynamical applications. *Acta Math.* 43 (1922), 1–119. [Reprinted in: *Collected Mathematical Papers*, Dover/AMS 1968].
- [Bi2] — Sur quelques courbes fermées remarquables. *Bull. Soc. Math. France* 60 (1932), 1–26. [Reprinted in: *Collected Mathematical Papers*, Dover/AMS 1968].
- [Fa] FATHI, A. Appendix to Chapter I of [He].
- [He] HERMAN, M. Sur les courbes invariantes par les difféomorphismes de l'anneau I. *Astérisque* 103–104 (1983).
- [KH] KATOK, A. and B. HASSELBLATT. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1995.
- [LCa] LE CALVEZ, P. Propriétés dynamiques des difféomorphismes de l'anneau et du tore. *Astérisque* 204 (1991).
- [MP] MACKAY, R. S. and I. C. PERCIVAL. Converse KAM: theory and practice. *Comm. Math. Phys.* 98 (1985), 469–512.
- [Ma1] MATHER, J. N. Glancing billiards. *Ergodic Theory Dynam. Systems* 2 (1982), 397–403.
- [Ma2] — Non-existence of invariant circles. *Ergodic Theory Dynam. Systems* 4 (1984), 301–309.
- [Ma3] — Variational construction of orbits for twist diffeomorphisms. *J. Amer. Math. Soc.* 4 (1991), 207–263.
- [MF] MATHER, J. N. and G. FORNI. Action minimizing orbits in Hamiltonian systems. In: S. Graffi (ed.): *Transition to Chaos in Classical and Quantum Mechanics*. Springer LNM 1589, 1992.