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Proof of (iii) *in the Theorem.* Finally, (iii) follows now immediately from the last inequality above, since we have:

$$\begin{aligned}\|\Phi_i(e_s(b_i))\|_{2,\mathrm{Tr}} &\geq \|\Phi_0^*(\Phi_i(e_s(b_i)))\|_{2,\mathrm{Tr}}\\ &\geq \|e_s(b_0)\|_{2,\mathrm{Tr}} - \|e_s(b_0) - \Phi_i'(e_s(b_i))\|_{2,\mathrm{Tr}}\\ &> (1-\delta'^{1/8})\|e_{t^{1/2}}(b_0)\|_{2,\mathrm{Tr}}\\ &> (1-\delta'^{1/8})(1+\delta^{1/16})^{-1/2}\|e_{t^{1/2}}(b_i)\|_{2,\mathrm{Tr}}\\ &\geq (1-\delta^{1/32})\|e_{t^{1/2}}(b_i)\|_{2,\mathrm{Tr}}\,.\end{aligned}$$

This ends the proof of the last part of the theorem. □

Proof of Corollary 0.2. As for the Corollary in the Introduction, it follows readily from the Theorem, by taking $n = 1$, $\Phi_0 = \Phi_1 = \Phi$, once we observe that, since Φ is positive, it is selfadjoint, so

$$\begin{aligned}&\sup\{\|\Phi(x)\|_{2,\mathrm{Tr}} \mid x \in P_1,\ \|x\|_{2,\mathrm{Tr}} = 1\}\\ &\qquad = \sup\{\|\Phi(x)\|_{2,\mathrm{Tr}} \mid x \in P_1,\ x = x^*,\ \|x\|_{2,\mathrm{Tr}} = 1\},\end{aligned}$$

and also by noticing that if $x \in P_1$ is such that $x = x^*$ then $\|\Phi(|x|)\|_{2,\mathrm{Tr}} \geq \|\Phi(x)\|_{2,\mathrm{Tr}}$. Indeed, this is because by approximating x by step functions (through spectral calculus) we may assume $x = \sum_i c_i p_i$ for some real scalars c_i and finitely many, mutually orthogonal projections of finite trace p_i. Then, taking into account that $\Phi(p_i), \Phi(p_j) \geq 0$ implies $\mathrm{Tr}(\Phi(p_i)\Phi(p_j)) \geq 0$, we get:

$$\begin{aligned}\|\Phi(x)\|_{2,\mathrm{Tr}}^2 &= \textstyle\sum_{i,j} c_i\bar{c}_j\,\mathrm{Tr}(\Phi(p_i)\Phi(p_j))\\ &\leq \textstyle\sum_{i,j} |c_i||\bar{c}_j|\,\mathrm{Tr}(\Phi(p_i)\Phi(p_j)) = \|\Phi(|x|)\|_{2,\mathrm{Tr}}^2\,.\end{aligned}$$

□

2. APPLICATIONS

We shall apply Theorem 0.1 to a case when the semifinite algebras are in fact commutative. We mention that the noncommutativity will be implicitly present though, through the consideration of the positive maps. Note also that in the proof of the Corollary below, only part (i) in the conclusion of the Theorem is being used. In turn, the proof of this part of the Theorem is relatively short.

COROLLARY 2.1. *Let $T = (t_{kk'})_{k,k' \in K}$ be a symmetric matrix with non-negative entries, only finitely many of which are non-zero on each row and column and with $t_{kk'} \geq 1$ whenever different from 0. Assume that for some $\alpha > 0$ and $\delta > 0$ the following conditions are satisfied:*

(a) *There exists a positive (possibly unbounded) function $v \colon K \to \mathbf{R}_+^*$ such that $Tv = \alpha v$.*

(b) *If we denote $\|T\| = \sup\{\|Tb\|_2 \mid b \in \ell^2(K),\ \|b\|_2 = 1\}$, then $\alpha \geq \|T\| > (1 - \delta^2/2)\alpha$, in which we denoted by $\| \quad \|_2$ the norm in $\ell^2(K)$.*

Then there exists a finite non-empty subset $F \subset K$ such that

$$\sum_{k \in \partial F} v_k^2 < (\alpha)^4 \delta^{1/4} \sum_{k \in F} v_k^2 ,$$

where $\partial F = \{k' \in K \setminus F \mid \exists k \in F \text{ with } t_{kk'} \neq 0\}$.

Before deriving 2.1 above from Theorem 0.1, let us point out right away a simple consequence of the hypothesis of 2.1, needed below, and which is in fact contained in the first 3 lines of the proof of 3.2 on page 281 of [Po3].

LEMMA 2.2. *Let $T = (t_{kk'})_{k,k' \in K}$ be a matrix with non-negative entries and only finitely many $t_{kk'} \neq 0$ on each row and column. Assume there exists $\alpha > 0$ and $v \colon K \to \mathbf{R}_+^*$ such that $Tv = \alpha v$. Then we have $\alpha v(k)/v(k') \geq t_{kk'}$, for all $k, k' \in K$.*

Proof. For each subset $S \subset K$ denote by $T(S) = \{k' \in K \mid \exists k \in S, \text{ with } t_{kk'} \neq 0\}$. Also, if $w \colon K \to \mathbf{C}$ then w_S denotes its restriction to S. With these notations we have $\alpha v_S = (Tv_{T(S)})_S$. Thus, if $k' \in T(S)$ and $k \in S$ is such that $t_{kk'} \neq 0$ then $\alpha v(k) \geq v(k') t_{kk'}$. If $t_{kk'} = 0$ there is nothing to prove. $\square$

Proof of 2.1. Let $\lambda = \alpha^{-1}$ and $\Phi = \lambda V T V^{-1}$, where V is the diagonal matrix over K with entries $v(k) = v_k$, $k \in K$. Note that Φ defines a bounded positive linear operator from $P \overset{\text{def}}{=} \ell^\infty(K)$ into itself such that $\Phi(1) = 1$. Let Tr denote the trace on P given by the weights $(v_k^2)_{k \in K}$ on K, i.e., if $b \in P = \ell^\infty(K)$ then

$$\|b\|_{1,\mathrm{Tr}} = \sum_{k \in K} |b_k| v_k^2 .$$

For $a, b \colon K \to \mathbf{C}$, at least one of which has finite support, we denote $\langle a, b \rangle = \sum_{k \in K} a_k \bar{b}_k$. For each $b \in P = \ell^\infty(K)$ with finite support we then have:

$$\begin{aligned}\mathrm{Tr}(\Phi(b)) &= \langle \Phi(b), V^2(1)\rangle = \langle b, \lambda V T V^{-1} V^2(1)\rangle \\ &= \langle b, \lambda V T V(1)\rangle = \langle b, V^2(1)\rangle = \mathrm{Tr}(b)\,.\end{aligned}$$

Thus $\mathrm{Tr}\circ\Phi = \mathrm{Tr}$. In particular, by Kadison's inequality, this implies $\|\Phi(a)\|_{2,\mathrm{Tr}} \leq \|a\|_{2,\mathrm{Tr}}$, $\forall a \in L^2(P, \mathrm{Tr})$.

Since $\|\lambda T\| > (1 - \delta^2/2)$, it follows that $\exists\ F_0 \subset K$ finite such that $T_0 =_{F_0} (\lambda T)_{F_0}$ satisfies $1 \geq \|T_0\| \geq 1-\delta^2/2$. By the classical Perron-Frobenius theorem applied to T_0 (which is a finite symmetric matrix with nonnegative entries) it follows that there exists $b_0 \in \ell^\infty(K) \simeq P$, supported in the set F_0, with $b_0(k) \geq 0$, $\forall k$, and $\langle b_0, b_0\rangle = 1$, such that $T_0 b_0 \geq (1 - \delta^2/2) b_0$. Thus, $\lambda T b_0 \geq (1 - \delta^2/2) b_0$.

Let then $b \overset{\mathrm{def}}{=} V^{-1}(b_0) \in \ell^\infty(K)$ and note that

$$\|b\|_{2,\mathrm{Tr}}^2 = \langle V^{-1}(b_0), V^2 V^{-1}(b_0)\rangle = \langle b_0, b_0\rangle = 1\,.$$

Moreover, we have:

$$\begin{aligned}\|\Phi(b) - b\|_{2,\mathrm{Tr}}^2 &\leq 2 - 2\,\mathrm{Tr}(\Phi(b) b) \\ &= 2 - 2\langle \lambda V^{-1} T(b_0), V(b_0)\rangle \\ &= 2 - 2\langle \lambda T(b_0), b_0\rangle \\ &\leq 2 - 2(1 - \delta^2/2) = 2\delta^2/2 = \delta^2\,.\end{aligned}$$

Thus $\|b - \Phi(b)\|_{2,\mathrm{Tr}} < \delta$ and $\|\Phi(b)\|_{2,\mathrm{Tr}} \geq 1 - \delta$, while $\|b\|_{2,\mathrm{Tr}} = 1$.

By Theorem 0.1 it follows that if $\delta < 5^{-32}$ then there exists a finite spectral projection e of b such that $\|\Phi(e) - e\|_{2,\mathrm{Tr}} < \delta^{1/4}\|e\|_{2,\mathrm{Tr}}$. Note that by approximating if necessary e in the norm $\|\ \ \|_{2,\mathrm{Tr}}$ by projections which are supported on finite subsets of K, we can obviously assume e itself is supported on a finite subset of K.

In particular we have:

$$\begin{aligned}\|(1 - e)\Phi(e)\|_{2,\mathrm{Tr}}^2 &\leq \|(1 - e)\Phi(e)\|_{2,\mathrm{Tr}}^2 + \|e - e\Phi(e)\|_{2,\mathrm{Tr}}^2 \\ &= \|e - \Phi(e)\|_{2,\mathrm{Tr}}^2 < \delta^{1/4}\|e\|_{2,\mathrm{Tr}}^2\,.\end{aligned}$$

Let $F \subset K$ be the support set of $e \in \ell^\infty(K) \simeq P$. By Lemma 2.2 we have $v_k^{-1} v_{k_0} \geq \lambda t_{kk_0}$ for all $k_0, k \in K$ for which $t_{kk_0} \neq 0$. Since $t_{kk_0} \geq 1$ for such k, k_0, we get $(\Phi)_{kk_0} = \lambda v_k^{-1} v_{k_0} t_{kk_0} \geq \lambda^2$, for all $k, k_0 \in K$ for which the entry (k, k_0) of Φ is nonzero. In particular, this shows that $\Phi(e)(1 - e) \geq \lambda^2 \chi_{\partial F}$, where $\chi_{\partial F} \in \ell^\infty(K)$ is the characteristic function of $\partial F \subset K$. Thus we have

$$\begin{aligned}\lambda^4 \sum_{k \in \partial F} v_k^2 &= \left\|\lambda^2 \chi_F\right\|_{2,\mathrm{Tr}}^2 \\ &\leq \left\|(1-e)\Phi(e)\right\|_{2,\mathrm{Tr}}^2 < \delta^{1/4}\|e\|_{2,\mathrm{Tr}}^2 \\ &= \delta^{1/4} \sum_{k \in F} v_k^2\end{aligned}$$

giving in the end the estimate:

$$\sum_{k \in \partial F} v_k^2 < \alpha^4 \delta^{1/4} \sum_{k \in F} v_k^2 ,$$

thus completing the proof. □

Corollary 2.3. *Let $\Gamma = (a_{kl})_{k \in K, l \in L}$ be a bipartite graph, with K and L labeling its even and respectively odd vertices and a_{kl} being the number of edges between the vertices k and l. Assume there exist $\alpha > 0$ and $\vec{v} = (v_k)_{k \in K}$, with $v_k > 0, \forall k \in K$ such that $\Gamma\Gamma^{\mathrm{t}}\vec{v} = \alpha\vec{v}$. Then Γ satisfies the Kesten-type amenability condition $\|\Gamma\|^2 = \alpha$ if and only if it satisfies the Følner-type condition:*

$$\forall \varepsilon > 0, \ \exists F \subset K, \text{ finite}, \ F \neq \varnothing, \ \textit{such that} \sum_{k \in \partial F} v_k^2 < \varepsilon \sum_{k \in F} v_k^2 .$$

Moreover, if this is the case, then Γ will satisfy the above Følner condition for any other weight vector $\vec{w} = (w_k)_k > 0$ with $\Gamma\Gamma^{\mathrm{t}}\vec{w} = \alpha\vec{w}$.

Proof. Simply apply 2.1 to $T = \Gamma\Gamma^{\mathrm{t}}$. Note that this statement can be easily derived from Corollary 0.2 in the introduction as well. □

Weighted bipartite graphs have become of particular interest in recent years due to their occurence in the Jones theory of subfactors of finite index ([J], [GHJ]). Thus, the consecutive inclusions of the higher relative commutants of a subfactor $N \subset M$ of finite index, $[M : N] < \infty$, are described by a pointed, bipartite graph $\Gamma_{N,M}$, called the *standard*, or *principal graph* of $N \subset M$. Moreover, $\Gamma_{N,M}$ has a canonical weight vector $\vec{v}$, given by the square roots of the local indices in the Jones tower, satisfying $\Gamma\Gamma^{\mathrm{t}}\vec{v} = [M : N]\vec{v}$, when $N \subset M$ satisfies a certain extremality conditions, and $\Gamma\Gamma^{\mathrm{t}}\vec{v} = [M : N]_{\min}\vec{v}$, in general, $[M : N]_{\min}$ being the minimal index for $N \subset M$ ([Hi]).

The amenability condition for such graphs, and more generally for arbitrary weighted bipartite graphs $\Gamma, \alpha, \vec{v}$, has been considered by the author in several papers and lectures starting in 1988, initially in the form of the Kesten type condition in 2.2 (see e.g., [Po1,2,4]). The Følner-type condition was first considered in [Po3] and the equivalence of the two conditions, for graphs of subfactors, was shown in [Po3,4] (see also [Po5] for an operatorial proof). Both these equivalent notions of amenability are important in the classification of subfactors ([Po1,2,3,5]). Thus, it has been proved that hyperfinite subfactors with amenable graph are completely classified by their higher relative commutants invariant (the *standard invariant*).

The above Corollary 2.3 shows that in fact the equivalence between the two notions of amenability holds true in a very general setting, for all bipartite graphs. This includes a more general class of graphs that appear in the theory of subfactors. To describe them, let us first note the following:

LEMMA 2.4. *Let* $N \subset M$ *be an extremal inclusion of type* II_1 *factors and assume* $Q \subset N$ *(respectively* $M \subset P$*) is a factor such that* $\dim(Q' \cap N) < \infty$ *(resp.* $\dim(M' \cap P) < \infty$*). Then the sequence of inclusions of finite dimensional algebras* $Q' \cap N \subset Q' \cap M \subset Q' \cap M_1 \subset \dots$ *(resp.* $M' \cap P \subset N' \cap P \subset N_1' \cap P \subset \dots$ *), in which* N_j, M_k *give a Jones tunnel-tower for* $N \subset M$*, with their corresponding traces, are described by a bipartite graph* Γ *with a weight vector* $\vec{t} = (t_k)_{k \in K}$ *such that* $\Gamma\Gamma^{\mathrm{t}}\vec{t} = [M:N]\vec{t}$.

Proof. The proof is identical to the proof of 1.7 in [Po6]. ☐

DEFINITION 2.5. Weighted bipartite graphs $\Gamma, \alpha, \vec{t}$ associated to an extremal subfactor $N \subset M$, with $\alpha = [M:N] < \infty$, and to a factor $Q \subset N$, with $\dim(Q' \cap N) < \infty$ (respectively $M \subset P$, with $\dim(M' \cap P) < \infty$), like in 2.4, are called *l-semi-standard graphs* (resp. *r-semi-standard graphs*). From 2.3 we can thus immediately infer:

COROLLARY 2.6. *A semi-standard graph* Γ *associated to a subfactor satisfies the Kesten-type condition* $\|\Gamma\|^2 = [M:N]$ *if and only if it satisfies the Følner-type condition:*

$$\forall \varepsilon > 0,\ \exists F \subset K,\ \text{finite},\ F \neq \emptyset,\ \textit{such that}\ \sum_{k \in \partial F} t_k^2 < \varepsilon \sum_{k \in F} t_k^2 .$$

Note added in proof. After this paper had been accepted for publication, we learned that A. Zuk had recently obtained a statement similar to the above Corollary 2.1, i.e., the equivalence between the Kesten and the Følner type amenability conditions for arbitrary weighted graphs (cf. Chapter 6 in

"Sur certaines propriétés spectrales du laplacien sur les graphes", University Paul Sabatier, Toulouse, thesis 1996). He proved this result by using different methods than ours. Note that Zuk's result generalized (unknowingly!) our previous similar statement which only covered the particular graphs coming from subfactors ([Po2,3,4]). On the other hand, our Corollary 0.2 in the present paper proves (by using Connes' distribution trick) an equivalence between Kesten and Følner type amenability conditions that is sensibly more general than all these prior results.

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