Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 45 (1999)

Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: HARMONIC ANALYSIS ON VECTOR BUNDLES OVER Sp(1,n)/Sp(1)xSp(n)

Autor: van Dijk, G. / PASQUALE, A.

Kapitel: 1. The fine structure of Sp(1,n)

DOI: https://doi.org/10.5169/seals-64447

In Section 6 we prove that the τ-Abel transform is an isomorphism of $\mathcal{D}(G;\chi_\tau)$ onto the convolution algebra $\mathcal{D}_+(\mathbf{R})$ of the even C^∞ compactly supported functions on $\mathbf{R}$. The inversion formula is explicitly written. The Paley-Wiener Theorem for the τ-spherical transform is an immediate consequence. The final Section 7 contains the inversion formula and the Plancherel Theorem for the τ-spherical transform.

Similar results for $\mathrm{SU}(n,1)$ have been obtained as a specialization of the Hermitian symmetric case by Shimeno [Shi] and Heckman [HS, Part 1].

ACKNOWLEDGMENT. During the preparation of this paper, the second author has been financially supported by the Dutch Organization for Scientific Research (N.W.O.).

1. THE FINE STRUCTURE OF $\mathrm{Sp}(1,n)$

Let $\mathbf{H}$ be the skew-field of the quaternions. Consider on the right $\mathbf{H}$-vector space $\mathbf{H}^{n+1}$ the Hermitian form

$$[x,y] = \bar{y}_0 x_0 - \bar{y}_1 x_1 - \cdots - \bar{y}_n x_n\,, \tag{1.1}$$

the bar sign denoting quaternionic conjugation: if $1, i, j, k$ are the quaternionic units and $q = a + ib + jc + kd \in \mathbf{H}$ (with $a,b,c,d \in \mathbf{R}$), then $\bar{q} = a - ib - jc - kd$. Let $G = \mathrm{Sp}(1,n)$ be the group $\mathrm{U}(1,n;\mathbf{H})$ of $(n+1)\times(n+1)$ matrices with coefficients in $\mathbf{H}$ which preserve this form. For $n=1$, G is called the De Sitter group. Let $\mathrm{Sp}(m)$ indicate the group $\mathrm{U}(m;\mathbf{H})$ of $m \times m$ matrices with coefficients in $\mathbf{H}$ which preserve the inner product $(x,y) = \bar{y}_1 x_1 + \cdots + \bar{y}_m x_m$ of $\mathbf{H}^m$. In particular, $\mathrm{Sp}(1)$ consists of the quaternions $q = a + i\,b + j\,c + k\,d$ with norm $|q| = \sqrt{a^2+b^2+c^2+d^2}$ equal to 1. $\mathrm{Sp}(1)$ is canonically isomorphic to $\mathrm{SU}(2)$. The group G acts on the projective space $P_n(\mathbf{H})$. Let Ω denote the image of the open set $\{x \in \mathbf{H}^{n+1} : [x,x] > 0\}$ under the canonical map $\mathbf{H}^{n+1} \setminus \{0\} \to P_n(\mathbf{H})$. Then G acts transitively on Ω, and the stabilizer of the quaternionic line generated by the vector $(1,0,\ldots,0)$ is the group

$$K = \left\{ \begin{bmatrix} u & 0 \\ 0 & U \end{bmatrix} : u \in \mathrm{Sp}(1),\, U \in \mathrm{Sp}(n) \right\} \equiv \mathrm{Sp}(1) \times \mathrm{Sp}(n)\,.$$

The homogeneous space G/K is called the hyperbolic quaternionic space. K is a maximally compact subgroup of G. G is connected and simply connected.

To study the fine structure of G, we consider its Lie algebra $\mathfrak{g} = \mathfrak{sp}(1,n)$. Let J be the $(n+1)\times(n+1)$ matrix $\operatorname{diag}(-1,1,\ldots,1)$. For any matrix X of type $(n+1, n+1)$ with coefficients in $\mathbf{H}$ we set $X^* = J\overline{X}^t J$, the symbol ^{t} denoting transposition.

The Lie algebra $\mathfrak{g}$ consists of the matrices X which verify the relation

$$X + X^* = 0.$$

These are the matrices of the form

$$\begin{bmatrix} Z_1 & Z_2 \\ \overline{Z}_2^t & Z_3 \end{bmatrix}$$

with Z_1 and Z_3 anti-Hermitian of type $(1,1)$ and (n,n), respectively, and Z_2 arbitrary. Let θ be the anti-involutive automorphism of $\mathfrak{g}$ defined by

$$\theta X = JXJ.$$

Then θ is a Cartan involution with the usual decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Here $\mathfrak{k}$ is the Lie algebra of K. Let L be the following element of $\mathfrak{g}$:

$$L = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \mathbf{0} & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then $L \in \mathfrak{p}$ and $\mathfrak{a} = \mathbf{R}L$ is a maximal Abelian subspace of $\mathfrak{p}$. We are going to diagonalize $\operatorname{ad} L$. The centralizer of L in $\mathfrak{k}$ is the subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ of the matrices

$$\begin{bmatrix} u & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & u \end{bmatrix}$$

with $u \in \mathbf{H}$, $u + \bar{u} = 0$ and V a matrix of type $(n-1, n-1)$ satisfying $V + \overline{V}^t = \mathbf{0}$. The non-zero eigenvalues of $\operatorname{ad} L$ are $\alpha = 1$, $-\alpha$, $\pm 2\alpha$. The space $\mathfrak{g}_\alpha$ consists of the matrices

$$X = \begin{bmatrix} 0 & z^* & 0 \\ z & \mathbf{0} & -z \\ 0 & z^* & 0 \end{bmatrix}$$

where z is a matrix of type $(n-1, 1)$ with coefficients in $\mathbf{H}$, and $z^* := \bar{z}^t$. The real dimension of $\mathfrak{g}_\alpha$ is $m_\alpha = 4(n-1)$. The space $\mathfrak{g}_{2\alpha}$ consists of the matrices of the form

$$X = \begin{bmatrix} w & 0 & -w \\ 0 & \mathbf{0} & 0 \\ w & 0 & -w \end{bmatrix}$$

with $w \in \mathbf{H}$, $w + \overline{w} = 0$. The dimension of $\mathfrak{g}_{2\alpha}$ is equal to $m_{2\alpha} = 3$. We have $\mathfrak{g} = \mathfrak{g}_{-2\alpha} + \mathfrak{g}_{-\alpha} + \mathfrak{m} + \mathfrak{a} + \mathfrak{g}_{\alpha} + \mathfrak{g}_{2\alpha}$.

Let A be the subgroup $\exp \mathfrak{a}$. This is the subgroup of the matrices

$$a_t = \begin{bmatrix} \cosh t & 0 & \sinh t \\ 0 & I & 0 \\ \sinh t & 0 & \cosh t \end{bmatrix}$$

where t is a real number. The centralizer of A in K is the subgroup M of the matrices

$$m(u, V) = \begin{bmatrix} u & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & u \end{bmatrix}$$

with $u \in \mathrm{Sp}(1)$ and $V \in \mathrm{Sp}(n-1)$. The Lie algebra of M is $\mathfrak{m}$. The subspace $\mathfrak{n} = \mathfrak{g}_{\alpha} + \mathfrak{g}_{2\alpha}$ is a (real) nilpotent subalgebra. Set $N = \exp \mathfrak{n}$. This is the subgroup of the matrices

$$n(w, z) = \begin{bmatrix} 1 + w - \frac{1}{2}[z, z] & z^* & -w + \frac{1}{2}[z, z] \\ z & I & -z \\ w - \frac{1}{2}[z, z] & z^* & 1 - w + \frac{1}{2}[z, z] \end{bmatrix}$$

where $w \in \mathbf{H}$ satisfies $w + \overline{w} = 0$ and $z = [z_1, \ldots, z_{n-1}]^t$ is a matrix of type $(n-1, 1)$ with coefficients in $\mathbf{H}$. We have set $z^* = \bar{z}^t$ and $[z, z] = -\bar{z}_1 z_1 - \cdots - \bar{z}_{n-1} z_{n-1}$.

The composition law in N is the following:

$$n(w, z) \cdot n(w', z') = n(w + w' + \Im[z, z'], z + z'),$$

where $\Im q := \frac{q - \bar{q}}{2}$ for $q \in \mathbf{H}$. The subgroups A and M normalize N:

$$a_t n(w, z) a_{-t} = n(e^{2t} w, e^t z),$$
$$m(u, V) n(w, z) m(u, V)^{-1} = n(u w \bar{u}, V z \bar{u}).$$

Let 2ρ be the trace of the restriction of $\operatorname{ad} L$ to $\mathfrak{n}$:

$$\rho = \frac{1}{2}(m_{\alpha} + 2m_{2\alpha}) = 2n + 1. \tag{1.2}$$

We have the Iwasawa decomposition $G = KAN = KNA$ and the corresponding integral formulas:

$$\int_G f(g)\, dg = \int_K \int_{-\infty}^{+\infty} \int_N f(k a_t n) e^{2\rho t}\, dk\, dt\, dn \tag{1.3}$$

$$= \int_K \int_N \int_{-\infty}^{+\infty} f(k n a_t)\, dk\, dn\, dt \tag{1.4}$$

for $f \in C_c(G)$. We adopt the usual notation $C_c(G)$ for the space of continuous functions on G with compact support. In the above formulas, $dn = dw\,dz$ $(n = n(w,z))$ and dk is the normalized Haar measure on K.

Let

$$K_1 = \left\{ \begin{bmatrix} u & 0 \\ 0 & I \end{bmatrix} : u \in \mathrm{Sp}(1) \right\}, \qquad K_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & U \end{bmatrix} : U \in \mathrm{Sp}(n) \right\}.$$

Then every $g \in G$ can be written as $g = k_1 k_2 a_t k_2'$ for uniquely determined $k_1 \in K_1$, $t \geq 0$ and for some $k_2, k_2' \in K_2$. Writing $g = [g_{ij}]_{i,j=0}^n$, we have

$$k_1 = \frac{g_{00}}{|g_{00}|} \quad \text{and} \quad \cosh t = |g_{00}|. \tag{1.5}$$

If $g \notin K$, then $t > 0$ and k_2, k_2' are uniquely determined modulo the subgroup

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & 1 \end{bmatrix} : V \in \mathrm{Sp}(n-1) \right\}.$$

After dg is normalized according to (1.3), the corresponding integral formula is

$$\int_G f(g)\,dg = \frac{1}{2}\left(\frac{\pi}{4}\right)^{2n} \frac{1}{\Gamma(2n)} \int_{K_1}\int_{K_2}\int_0^\infty\int_{K_2} f(k_1 k_2 a_t k_2')\Delta(t)\,dk_1\,dk_2\,dt\,dk_2' \tag{1.6}$$

where

$$\Delta(t) := 2^{2\rho}(\sinh t)^{4n-1}(\cosh t)^3. \tag{1.7}$$

2. THE CONVOLUTION ALGEBRA $\mathcal{D}(G;\chi_l)$

Let $\mathbf{N}/2$ be the set of nonnegative half-integers $\{0, 1/2, 1, 3/2, \dots\}$. Since $K_1 \equiv \mathrm{Sp}(1)$ is isomorphic to $\mathrm{SU}(2)$, $\mathbf{N}/2$ parametrizes the set of equivalence classes of unitary irreducible representations of K_1. We denote with the same symbol τ_l either the equivalence class corresponding to the parameter l or a fixed representative for it. Thus τ_l is a unitary irreducible representation of K_1 in a Hilbert space V_l of dimension $d_l = 2l+1$. We extend τ_l to a representation of K by setting $\tau_l \equiv \mathbf{1}$ on K_2. Each τ_l is self-dual, i.e. unitarily equivalent to its contragredient representation. It follows in particular that the character $\chi_l = \operatorname{tr}\tau_l$ of τ_l satisfies $\chi_l(k^{-1}) = \chi_l(k)$, $k \in K$.

We denote by $\mathcal{D}(G;\tau_l)$ the convolution algebra of the compactly supported C^∞ maps $F\colon G \to \mathrm{End}(V_l)$ satisfying