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**Artikel:** HARMONIC ANALYSIS ON VECTOR BUNDLES OVER  $Sp(1,n)/Sp(1) \times Sp(n)$   
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for  $f \in C_c(G)$ . We adopt the usual notation  $C_c(G)$  for the space of continuous functions on  $G$  with compact support. In the above formulas,  $dn = dw dz$  ( $n = n(w, z)$ ) and  $dk$  is the normalized Haar measure on  $K$ .

Let

$$K_1 = \left\{ \begin{bmatrix} u & 0 \\ 0 & I \end{bmatrix} : u \in \mathrm{Sp}(1) \right\}, \quad K_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & U \end{bmatrix} : U \in \mathrm{Sp}(n) \right\}.$$

Then every  $g \in G$  can be written as  $g = k_1 k_2 a_t k'_2$  for uniquely determined  $k_1 \in K_1$ ,  $t \geq 0$  and for some  $k_2, k'_2 \in K_2$ . Writing  $g = [g_{ij}]_{i,j=0}^n$ , we have

$$(1.5) \quad k_1 = \frac{g_{00}}{|g_{00}|} \quad \text{and} \quad \cosh t = |g_{00}|.$$

If  $g \notin K$ , then  $t > 0$  and  $k_2, k'_2$  are uniquely determined modulo the subgroup

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & 1 \end{bmatrix} : V \in \mathrm{Sp}(n-1) \right\}.$$

After  $dg$  is normalized according to (1.3), the corresponding integral formula is

$$(1.6) \quad \int_G f(g) dg = \frac{1}{2} \left(\frac{\pi}{4}\right)^{2n} \frac{1}{\Gamma(2n)} \int_{K_1} \int_{K_2} \int_0^\infty \int_{K_2} f(k_1 k_2 a_t k'_2) \Delta(t) dk_1 dk_2 dt dk'_2$$

where

$$(1.7) \quad \Delta(t) := 2^{2\rho} (\sinh t)^{4n-1} (\cosh t)^3.$$

## 2. THE CONVOLUTION ALGEBRA $\mathcal{D}(G; \chi_l)$

Let  $\mathbf{N}/2$  be the set of nonnegative half-integers  $\{0, 1/2, 1, 3/2, \dots\}$ . Since  $K_1 \cong \mathrm{Sp}(1)$  is isomorphic to  $\mathrm{SU}(2)$ ,  $\mathbf{N}/2$  parametrizes the set of equivalence classes of unitary irreducible representations of  $K_1$ . We denote with the same symbol  $\tau_l$  either the equivalence class corresponding to the parameter  $l$  or a fixed representative for it. Thus  $\tau_l$  is a unitary irreducible representation of  $K_1$  in a Hilbert space  $V_l$  of dimension  $d_l = 2l + 1$ . We extend  $\tau_l$  to a representation of  $K$  by setting  $\tau_l \equiv \mathbf{1}$  on  $K_2$ . Each  $\tau_l$  is self-dual, i.e. unitarily equivalent to its contragredient representation. It follows in particular that the character  $\chi_l = \mathrm{tr} \tau_l$  of  $\tau_l$  satisfies  $\chi_l(k^{-1}) = \chi_l(k)$ ,  $k \in K$ .

We denote by  $\mathcal{D}(G; \tau_l)$  the convolution algebra of the compactly supported  $C^\infty$  maps  $F: G \rightarrow \mathrm{End}(V_l)$  satisfying

$$(2.8) \quad F(kxk') = \tau_l(k)F(x)\tau_l(k') \quad (k, k' \in K, x \in G).$$

Let  $\mathcal{D}(G)$  be the convolution algebra of the  $C^\infty$  compactly supported complex valued functions on  $G$ . Then  $\mathcal{D}(G; \tau_l)$  is isomorphic to the subalgebra  $\mathcal{D}(G; \chi_l)$  of  $\mathcal{D}(G)$  consisting of the functions  $f \in \mathcal{D}(G)$  satisfying

$$(2.9) \quad f^0 = f$$

and

$$(2.10) \quad f * d_l \chi_l = f,$$

where

$$(2.11) \quad f^0(x) := \int_K f(kxk^{-1}) dk.$$

The isomorphism is given by  $F \mapsto d_l \operatorname{tr} F$  (see e.g. [Dij], Theorem 1.1).

The commutativity of the algebra  $\mathcal{D}(G; \chi_l)$  can be deduced from the fact that the restriction  $\tau_l|_M$  of  $\tau_l$  to  $M$  is multiplicity free (according to the general criterion by Deitmar, cf. [Dei] Theorem 3, the commutativity of  $\mathcal{D}(G; \chi_l)$  and the multiplicity freeness of  $\tau_l|_M$  are in fact equivalent). An elementary direct argument by Takahashi proves the commutativity of a convolution algebra which is slightly bigger than  $\mathcal{D}(G; \chi_l)$ . Let  $\mathcal{D}_1(G)$  be the subalgebra of  $\mathcal{D}(G)$  consisting of the functions  $f \in \mathcal{D}(G)$  satisfying

$$(2.12) \quad f(k_2 k_1 g k_1^{-1} k_2') = f(g), \quad g \in G, k_1 \in K_1, k_2 \in K_2.$$

Clearly  $\mathcal{D}(G; \chi_l) \subset \mathcal{D}_1(G)$ . Moreover  $\mathcal{D}_1(G) = \oplus_l \mathcal{D}(G; \chi_l)$ . Showing that

$$(2.13) \quad f(g^{-1}) = f(g) \quad \text{for all } f \in \mathcal{D}_1(G) \text{ and } g \in G,$$

one proves the following lemma.

2.1. LEMMA ([T2], Proposition 1). *The algebra  $\mathcal{D}_1(G)$  is commutative.*

2.2. LEMMA (cf. [T2], Lemma 2). *For every function  $f \in \mathcal{D}(G; \chi_l)$ ,*

$$(2.14) \quad f(g) = f(k_1 a_t) = \frac{1}{d_l} \chi_l(k_1) f(a_t)$$

*if  $g = k_1 k_2 a_t k_2'$  ( $k_1 \in K_1; k_2, k_2' \in K_2; t \in \mathbf{R}$ ).*

*Proof.* For  $F \in \mathcal{D}(G; \tau_l)$ ,  $F(a_t)$  is a scalar multiple of the identity, since it commutes with  $\tau_l(k_1)$  for all  $k_1 \in K_1$ .  $\square$

2.3. REMARK. Formula (2.13) and Lemma 2.2 remain true for all continuous functions  $f$  such that  $f = f^0$ ,  $f * d_l \chi_l = f$ .