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Kapitel: 2. The convolution algebra $D(G; \lambda_l)$

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for $f \in C_c(G)$. We adopt the usual notation $C_c(G)$ for the space of continuous functions on G with compact support. In the above formulas, $dn = dw\,dz$ ($n = n(w,z)$) and dk is the normalized Haar measure on K.

Let

$$K_1 = \left\{ \begin{bmatrix} u & 0 \\ 0 & I \end{bmatrix} : u \in \mathrm{Sp}(1) \right\}, \qquad K_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & U \end{bmatrix} : U \in \mathrm{Sp}(n) \right\}.$$

Then every $g \in G$ can be written as $g = k_1 k_2 a_t k_2'$ for uniquely determined $k_1 \in K_1$, $t \geq 0$ and for some $k_2, k_2' \in K_2$. Writing $g = [g_{ij}]_{i,j=0}^{n}$, we have

$$k_1 = \frac{g_{00}}{|g_{00}|} \quad \text{and} \quad \cosh t = |g_{00}|. \tag{1.5}$$

If $g \notin K$, then $t > 0$ and k_2, k_2' are uniquely determined modulo the subgroup

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & 1 \end{bmatrix} : V \in \mathrm{Sp}(n-1) \right\}.$$

After dg is normalized according to (1.3), the corresponding integral formula is

$$\int_G f(g)\,dg = \frac{1}{2}\left(\frac{\pi}{4}\right)^{2n} \frac{1}{\Gamma(2n)} \int_{K_1}\int_{K_2}\int_0^\infty\int_{K_2} f(k_1 k_2 a_t k_2')\Delta(t)\,dk_1\,dk_2\,dt\,dk_2' \tag{1.6}$$

where

$$\Delta(t) := 2^{2\rho}(\sinh t)^{4n-1}(\cosh t)^3. \tag{1.7}$$

2. THE CONVOLUTION ALGEBRA $\mathcal{D}(G;\chi_l)$

Let $\mathbf{N}/2$ be the set of nonnegative half-integers $\{0, 1/2, 1, 3/2, \dots\}$. Since $K_1 \equiv \mathrm{Sp}(1)$ is isomorphic to $\mathrm{SU}(2)$, $\mathbf{N}/2$ parametrizes the set of equivalence classes of unitary irreducible representations of K_1. We denote with the same symbol τ_l either the equivalence class corresponding to the parameter l or a fixed representative for it. Thus τ_l is a unitary irreducible representation of K_1 in a Hilbert space V_l of dimension $d_l = 2l+1$. We extend τ_l to a representation of K by setting $\tau_l \equiv \mathbf{1}$ on K_2. Each τ_l is self-dual, i.e. unitarily equivalent to its contragredient representation. It follows in particular that the character $\chi_l = \operatorname{tr}\tau_l$ of τ_l satisfies $\chi_l(k^{-1}) = \chi_l(k)$, $k \in K$.

We denote by $\mathcal{D}(G;\tau_l)$ the convolution algebra of the compactly supported C^∞ maps $F\colon G \to \mathrm{End}(V_l)$ satisfying

$$F(kxk') = \tau_l(k)F(x)\tau_l(k') \qquad (k,k' \in K, x \in G). \tag{2.8}$$

Let $\mathcal{D}(G)$ be the convolution algebra of the C^∞ compactly supported complex valued functions on G. Then $\mathcal{D}(G;\tau_l)$ is isomorphic to the subalgebra $\mathcal{D}(G;\chi_l)$ of $\mathcal{D}(G)$ consisting of the functions $f \in \mathcal{D}(G)$ satisfying

$$f^0 = f \tag{2.9}$$

and

$$f * d_l\chi_l = f, \tag{2.10}$$

where

$$f^0(x) := \int_K f(kxk^{-1})\, dk. \tag{2.11}$$

The isomorphism is given by $F \longmapsto d_l \operatorname{tr} F$ (see e.g. [Dij], Theorem 1.1).

The commutativity of the algebra $\mathcal{D}(G;\chi_l)$ can be deduced from the fact that the restriction $\tau_l|_M$ of τ_l to M is multiplicity free (according to the general criterion by Deitmar, cf. [Dei] Theorem 3, the commutativity of $\mathcal{D}(G;\chi_l)$ and the multiplicity freeness of $\tau_l|_M$ are in fact equivalent). An elementary direct argument by Takahashi proves the commutativity of a convolution algebra which is slightly bigger than $\mathcal{D}(G;\chi_l)$. Let $\mathcal{D}_1(G)$ be the subalgebra of $\mathcal{D}(G)$ consisting of the functions $f \in \mathcal{D}(G)$ satisfying

$$f(k_2k_1gk_1^{-1}k_2') = f(g), \qquad g \in G, k_1 \in K_1, k_2 \in K_2. \tag{2.12}$$

Clearly $\mathcal{D}(G;\chi_l) \subset \mathcal{D}_1(G)$. Moreover $\mathcal{D}_1(G) = \oplus_l \mathcal{D}(G;\chi_l)$. Showing that

$$f(g^{-1}) = f(g) \qquad \text{for all } f \in \mathcal{D}_1(G) \text{ and } g \in G, \tag{2.13}$$

one proves the following lemma.

2.1. LEMMA ([T2], Proposition 1). *The algebra $\mathcal{D}_1(G)$ is commutative.*

2.2. LEMMA (cf. [T2], Lemma 2). *For every function $f \in \mathcal{D}(G;\chi_l)$,*

$$f(g) = f(k_1a_t) = \frac{1}{d_l}\chi_l(k_1)f(a_t) \tag{2.14}$$

if $g = k_1k_2a_tk_2'$ $(k_1 \in K_1; k_2, k_2' \in K_2;\ t \in \mathbf{R})$.

Proof. For $F \in \mathcal{D}(G;\tau_l)$, $F(a_t)$ is a scalar multiple of the identity, since it commutes with $\tau_l(k_1)$ for all $k_1 \in K_1$. □

2.3. REMARK. Formula (2.13) and Lemma 2.2 remain true for all continuous functions f such that $f = f^0$, $f * d_l\chi_l = f$.