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3. THE τ_l -SPHERICAL FUNCTIONS

Let $\mathfrak{U}(\mathfrak{g})$ be the universal enveloping algebra of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g} = \mathfrak{sp}(1, n)$. The elements of $\mathfrak{U}(\mathfrak{g})$ are considered as left-invariant differential operators on $G = \mathrm{Sp}(1, n)$ acting on C^∞ functions on the right:

$$f(g; X) := \left. \frac{d}{dt} f(g \exp tX) \right|_{t=0} \quad (f \in C^\infty(G), X \in \mathfrak{g}, g \in G).$$

We adopt Harish-Chandra's notation $f(g; D)$ for the image of $f \in C^\infty(G)$ under the right action of $D \in \mathfrak{U}(\mathfrak{g})$. The set of K -invariant elements of $\mathfrak{U}(\mathfrak{g})$ is denoted by $\mathfrak{U}(\mathfrak{g})^K$.

3.1. THEOREM ([Go], Theorems 8, 10 and 14; [GaV], Theorems 1.3.14, 1.4.5 and Proposition 1.4.4). *Let $l \in \mathbf{N}/2$ be fixed. Let ζ be a complex-valued continuous function on G satisfying (2.9), (2.10) and $\zeta(e) = 1$ (e is the unit element of G). The following statements are mutually equivalent.*

1. *The mapping $f \mapsto \int_G f(g) \zeta(g) dg$ is an algebra homomorphism of $\mathcal{D}(G; \chi_l)$ into \mathbb{C} .*

2. *ζ satisfies the functional equation*

$$(3.15) \quad \int_K \zeta(kg_1k^{-1}g_2) dk = \zeta(g_1)\zeta(g_2)$$

for all $g_1, g_2 \in G$.

3. *ζ is a common eigenfunction of the elements of $\mathfrak{U}(\mathfrak{g})^K$.*

A function ζ satisfying the equivalent conditions of Theorem 3.1 is called a *spherical function of type τ_l (and height 1)* or briefly a *τ_l -spherical function*¹⁾.

Observe that Condition 3 implies in particular that every τ_l -spherical function is analytic on G because $\mathfrak{U}(\mathfrak{g})^K$ contains an elliptic differential operator.

¹⁾ Our definition of τ_l -spherical function ζ is formally less restrictive than Godement's, which also requires ζ to be quasi-bounded with respect to some seminorm on G (cf. [Go], p. 519). It is possible to show (e.g. [GaV], Theorem 1.3.14) that each of the conditions for ζ given by Theorem 3.1 is equivalent to the existence of an irreducible Fréchet representation (T, \mathcal{H}) – with a d_l -dimensional K -isotypic subspace $\mathcal{H}(\tau_l)$ of type τ_l – for which

$$(*) \quad \zeta = \frac{1}{d_l} \mathrm{tr}[E(\tau_l)TE(\tau_l)],$$

$E(\tau_l)$ being the projection of \mathcal{H} onto $\mathcal{H}(\tau_l)$. (T, \mathcal{H}) can be chosen to be a Banach representation of G if and only if ζ is quasi-bounded with respect to some seminorm on G . In Section 5 we will determine, for each τ_l -spherical function ζ , an irreducible *Hilbert* representation (T, \mathcal{H}) for which $(*)$ holds. It follows, in particular, that the condition of quasi-boundedness is, in our case, automatically satisfied.

For complex-valued functions f on G and F on \mathbf{R} , we set

$$\begin{aligned} f^*(g) &= \overline{f(g^{-1})} & (g \in G) \\ F^*(t) &= \overline{F(-t)} & (t \in \mathbf{R}). \end{aligned}$$

The τ_l -Abel transform of $f \in \mathcal{D}(G; \chi_l)$ is the function $\mathcal{A}_l f$ on \mathbf{R} defined by

$$(3.16) \quad \mathcal{A}_l f(t) = \frac{1}{d_l^2} e^{\rho t} \int_N f(a_t n) dn.$$

Its properties are summarized in the following proposition.

3.2. PROPOSITION. *For all $f \in \mathcal{D}(G; \chi_l)$, $\mathcal{A}_l f$ is a C^∞ function on \mathbf{R} with compact support. If $f, f_1, f_2 \in \mathcal{D}(G; \chi_l)$ and $a_1, a_2 \in \mathbf{C}$, then*

$$(3.17) \quad (\mathcal{A}_l f)^* = \mathcal{A}_l(f^*),$$

$$(3.18) \quad \mathcal{A}_l(a_1 f_1 + a_2 f_2) = a_1 \mathcal{A}_l f_1 + a_2 \mathcal{A}_l f_2,$$

$$(3.19) \quad \mathcal{A}_l(f_1 * f_2) = \mathcal{A}_l f_1 * \mathcal{A}_l f_2.$$

Formula (3.17) is equivalent to the fact that $\mathcal{A}_l f$ is an even function.

Proof. Formulas (3.17)–(3.19) are immediately proven by passing to $\mathcal{D}(G; \tau_l)$. For the last statement, recall that $f(g^{-1}) = f(g)$ for $f \in \mathcal{D}(G; \chi_l)$. \square

The following lemma relates our definition of \mathcal{A}_l to the definition often found in the literature (cf. e.g. [W2], p. 34).

3.3. LEMMA. *For $f \in \mathcal{D}(G; \chi_l)$ one has*

$$\int_N f(ka_t n) dn = \frac{1}{d_l} \chi_l(k) \int_N f(a_t n) dn.$$

Proof. Let $F \in \mathcal{D}(G; \tau_l)$. Then $\int_N F(a_t n) dn$ commutes with $\tau(m)$ ($m \in M$) so with $\tau(k_1)$ ($k_1 \in K_1$), hence is a scalar multiple of the identity. The lemma follows by taking traces. \square

We now use the τ_l -Abel transform to construct τ_l -spherical functions. Because of Proposition 3.2, for any complex number s , the map

$$(3.20) \quad \lambda_s : f \longmapsto \int_{-\infty}^{\infty} \mathcal{A}_l f(t) e^{-st} dt$$

is an algebra homomorphism of $\mathcal{D}(G; \chi_l)$ into \mathbf{C} .

Set

$$(3.21) \quad \alpha_{l,s}(ka_t n) = \frac{1}{d_l} \chi_l(k) e^{-(s+\rho)t}.$$

Since $f = f * d_l \chi_l$ and $\chi_l(k^{-1}) = \chi_l(k)$ for $k \in K$, for every $f \in \mathcal{D}(G; \chi_l)$

$$\begin{aligned} \lambda_s(f) &= \frac{1}{d_l} \int_K \int_{-\infty}^{\infty} \int_N f(ka_t n) \chi_l(k) e^{(-s+\rho)t} dk dt dn \\ &= \int_G f(g) \alpha_{l,s}(g) dg \\ &= \int_G f(g) \int_K \alpha_{l,s}(kgk^{-1}) dk dg \\ (3.22) \quad &= \int_G f(g) \zeta_{l,s}(g) dg \end{aligned}$$

with

$$(3.23) \quad \zeta_{l,s} := \int_K \alpha_{l,s}(kgk^{-1}) dk.$$

One easily checks that $\zeta_{l,s}$ satisfies $\zeta_{l,s} = \zeta_{l,s}^0$, $\zeta_{l,s} * d_l \chi_l = \zeta_{l,s}$ and $\zeta_{l,s}(e) = 1$. Thus $\zeta_{l,s}$ is a τ_l -spherical function. It will be shown in the next section that any τ_l -spherical function is of the form (3.24).

By Remark 2.3, we have

$$(3.24) \quad \zeta_{l,s}(g) = \frac{1}{d_l} \chi_l(k_1) \zeta_{l,s}(a_t) \quad \text{for } g = k_1 k_2 a_t k_2',$$

so $\zeta_{l,s}$ is uniquely determined by its restriction to A .

4. THE DIFFERENTIAL EQUATION FOR THE τ_l -SPHERICAL FUNCTIONS

For a subalgebra \mathfrak{u} of \mathfrak{g} , let $\mathfrak{u}_{\mathbf{C}}$ denote the complex subalgebra of $\mathfrak{g}_{\mathbf{C}}$ generated by \mathfrak{u} . The universal enveloping algebra $\mathfrak{U}(\mathfrak{u})$ of $\mathfrak{u}_{\mathbf{C}}$ is considered as a subalgebra of $\mathfrak{U}(\mathfrak{g})$.

The representation τ_l of K_1 induces differentiated representations of the Lie algebra \mathfrak{k}_1 of K_1 and of the universal enveloping algebra $\mathfrak{U}(\mathfrak{k}_1)$ of $(\mathfrak{k}_1)_{\mathbf{C}}$. We indicate these representations with the same letter τ_l . Let \mathfrak{k}_2 be the Lie algebra of K_2 . Every element $Y \in \mathfrak{k}_{\mathbf{C}}$ can be uniquely decomposed as