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is an algebra homomorphism of  $\mathcal{D}(G; \chi_l)$  into  $\mathbf{C}$ .

Set

$$(3.21) \quad \alpha_{l,s}(ka_t n) = \frac{1}{d_l} \chi_l(k) e^{-(s+\rho)t}.$$

Since  $f = f * d_l \chi_l$  and  $\chi_l(k^{-1}) = \chi_l(k)$  for  $k \in K$ , for every  $f \in \mathcal{D}(G; \chi_l)$

$$(3.22) \quad \begin{aligned} \lambda_s(f) &= \frac{1}{d_l} \int_K \int_{-\infty}^{\infty} \int_N f(ka_t n) \chi_l(k) e^{(-s+\rho)t} dk dt dn \\ &= \int_G f(g) \alpha_{l,s}(g) dg \\ &= \int_G f(g) \int_K \alpha_{l,s}(kgk^{-1}) dk dg \\ &= \int_G f(g) \zeta_{l,s}(g) dg \end{aligned}$$

with

$$(3.23) \quad \zeta_{l,s} := \int_K \alpha_{l,s}(kgk^{-1}) dk.$$

One easily checks that  $\zeta_{l,s}$  satisfies  $\zeta_{l,s} = \zeta_{l,s}^0$ ,  $\zeta_{l,s} * d_l \chi_l = \zeta_{l,s}$  and  $\zeta_{l,s}(e) = 1$ . Thus  $\zeta_{l,s}$  is a  $\tau_l$ -spherical function. It will be shown in the next section that any  $\tau_l$ -spherical function is of the form (3.24).

By Remark 2.3, we have

$$(3.24) \quad \zeta_{l,s}(g) = \frac{1}{d_l} \chi_l(k_1) \zeta_{l,s}(a_t) \quad \text{for } g = k_1 k_2 a_t k_2',$$

so  $\zeta_{l,s}$  is uniquely determined by its restriction to  $A$ .

#### 4. THE DIFFERENTIAL EQUATION FOR THE $\tau_l$ -SPHERICAL FUNCTIONS

For a subalgebra  $\mathfrak{u}$  of  $\mathfrak{g}$ , let  $\mathfrak{u}_{\mathbf{C}}$  denote the complex subalgebra of  $\mathfrak{g}_{\mathbf{C}}$  generated by  $\mathfrak{u}$ . The universal enveloping algebra  $\mathfrak{U}(\mathfrak{u})$  of  $\mathfrak{u}_{\mathbf{C}}$  is considered as a subalgebra of  $\mathfrak{U}(\mathfrak{g})$ .

The representation  $\tau_l$  of  $K_1$  induces differentiated representations of the Lie algebra  $\mathfrak{k}_1$  of  $K_1$  and of the universal enveloping algebra  $\mathfrak{U}(\mathfrak{k}_1)$  of  $(\mathfrak{k}_1)_{\mathbf{C}}$ . We indicate these representations with the same letter  $\tau_l$ . Let  $\mathfrak{k}_2$  be the Lie algebra of  $K_2$ . Every element  $Y \in \mathfrak{k}_{\mathbf{C}}$  can be uniquely decomposed as

$Y = Y^{(1)} + Y^{(2)}$  with  $Y^{(j)} \in (\mathfrak{k}_j)_{\mathbb{C}}$ ,  $j = 1, 2$ . The symbol  $\chi_l$  will also be used for the  $\mathbb{C}$ -linear map on  $\mathfrak{U}(\mathfrak{k})$  defined by

$$\chi_l(Y_1 \cdots Y_m) := \text{tr} \left[ \tau_l(Y_1^{(1)}) \cdots \tau_l(Y_m^{(1)}) \right]$$

for  $Y_1, \dots, Y_m \in \mathfrak{k}_{\mathbb{C}}$ .

The Iwasawa decomposition  $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$  gives  $\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{k})\mathfrak{U}(\mathfrak{a})\mathfrak{U}(\mathfrak{n}) \cong \mathfrak{U}(\mathfrak{k}) \otimes \mathfrak{U}(\mathfrak{a}) \oplus \mathfrak{U}(\mathfrak{g})\mathfrak{n}_{\mathbb{C}}$ . Let  $P: \mathfrak{U}(\mathfrak{g}) \rightarrow \mathfrak{U}(\mathfrak{k}) \otimes \mathfrak{U}(\mathfrak{a})$  be the corresponding projection. For  $s \in \mathbb{C}$ , let  $e_s$  be the  $\mathbb{C}$ -linear map on  $\mathfrak{U}(\mathfrak{a})$  defined by

$$e_s(L^m) := (-1)^m (s + \rho)^m \quad \text{for every integer } m \geq 0.$$

Define  $p_{l,s}: \mathfrak{U}(\mathfrak{g}) \rightarrow \mathbb{C}$  to be the composition  $p_{l,s} := \left( \frac{1}{d_l} \chi_l \otimes e_s \right) \circ P$ , where as before  $d_l = \dim \tau_l$ .

4.1. PROPOSITION. *Let  $\zeta_{l,s}$  be the function defined by Formula (3.23). For every  $D \in \mathfrak{U}(\mathfrak{g})^K$  and  $g \in G$*

$$(4.25) \quad \zeta_{l,s}(g; D) = p_{l,s}(D) \zeta_{l,s}(g).$$

*Proof.* Because of Theorem 3.1,  $\zeta_{l,s}$  is an eigenfunction of every  $D \in \mathfrak{U}(\mathfrak{g})^K$ . The eigenvalue corresponding to  $D \in \mathfrak{U}(\mathfrak{g})^K$  is  $\zeta_{l,s}(e; D)$  because  $\zeta_{l,s}(e) = 1$ . Since  $D$  is  $K$ -invariant,  $\zeta_{l,s}(e; D) = \alpha_{l,s}(e; D)$ . Write  $D = \sum_i y_i x_i + \sum_j n_j$  with  $y_i \in \mathfrak{U}(\mathfrak{k})$ ,  $x_i \in \mathfrak{U}(\mathfrak{a})$  and  $n_j \in \mathfrak{U}(\mathfrak{g})\mathfrak{n}_{\mathbb{C}}$ . Then  $\alpha_{l,s}(e; D) = \sum_i \alpha_{l,s}(e; y_i x_i)$  because  $\alpha_{l,s}(gn) = \alpha_{l,s}(g)$  for  $g \in G$  and  $n \in N$ . To compute  $\alpha_{l,s}(e; y_i x_i)$ , assume without loss of generality that  $x_i = L^{m_i}$  and that  $y_i = Y_1 \cdots Y_m$  with  $Y_j \in \mathfrak{k}$ . The definition of  $\alpha_{l,s}$  gives

$$\alpha_{l,s}(e; y_i x_i) = \frac{1}{d_l} \chi_l(y_i) (-1)^m (s + \rho)^m = p_{l,s}(y_i x_i).$$

Thus  $\zeta_{l,s}(e; D) = p_{l,s}(D)$ .  $\square$

Let  $\delta_l(D)$  denote the  $\tau_l$ -radial component on  $A^+ := \{a_t : t > 0\}$  of the differential operator  $D \in \mathfrak{U}(\mathfrak{g})$ ; that is, the unique differential operator on  $A^+$  satisfying

$$f(a_t; \delta_l(D)) = f(a_t; D)$$

for all  $f \in \mathcal{D}(G; \chi_l)$  and  $t > 0$ . Proposition 4.1 immediately implies

4.2. COROLLARY.  *$\zeta_{l,s}$  is an eigenfunction of the  $\tau_l$ -radial component on  $A^+$  of every  $K$ -invariant differential operator on  $G$ :*

$$(4.26) \quad \zeta_{l,s}(a_t; \delta_l(D)) = p_{l,s}(D) \zeta_{l,s}(a_t) \quad (D \in \mathfrak{U}(\mathfrak{g})^K, t > 0).$$

We now write (4.26) explicitly in the case  $D$  is the Casimir operator  $\omega$  of  $\mathfrak{g}$ . Let  $B$  denote the Cartan-Killing form of  $\mathfrak{g}_{\mathbf{C}}$  ( $\cong \mathfrak{sp}(1+n, \mathbf{C})$ ). If  $X, Y \in \mathfrak{sp}(1, n)$ , then

$$B(X, Y) = 4(n + 2) \Re \text{tr}(XY)$$

where  $\Re$  denotes the quaternionic real part:  $\Re q = \frac{q+\bar{q}}{2}$  for  $q \in \mathbf{H}$ . The bilinear form  $B_{\theta}(X, Y) := -B(X, \theta Y)$  is an inner product on  $\mathfrak{g}$ . Orthonormality will be considered with respect to  $B_{\theta}$ .

Let  $\{Z_j\}_{j=1}^m$  ( $m := 2n^2 + n$ ) and  $\{X_{\beta,j}\}_{j=1}^{m_{\beta}}$  ( $\beta \in \{\alpha, 2\alpha\}$ ) be orthonormal bases in  $\mathfrak{m}$  and in  $\mathfrak{g}_{\beta}$  respectively. Define  $X_{-\beta,j} = -\theta(X_{\beta,j})$  for  $\beta \in \{\alpha, 2\alpha\}$  and  $j = 1, \dots, m_{\beta}$ . Then  $\{X_{-\beta,j}\}_{j=1}^{m_{\beta}}$  is an orthonormal basis for  $\mathfrak{g}_{-\beta}$ , and  $B(X_{\beta,i}, X_{-\beta,j}) = \delta_{ij}$ . Moreover, for all  $j = 1, \dots, m_{\beta}$ ,  $H_{\beta} := [X_{\beta,j}, X_{-\beta,j}]$  is the unique element of  $\mathfrak{a}$  satisfying  $B(H_{\beta}, L) = \beta(L)$ , i.e.

$$H_{\beta} = \frac{h_{\beta}}{8(n+2)} L \quad \text{with} \quad h_{\beta} = \begin{cases} 1 & \text{if } \beta = \alpha \\ 2 & \text{if } \beta = 2\alpha. \end{cases}$$

Set  $H_1 := \frac{L}{\sqrt{8(n+2)}}$ , a unit vector in  $\mathfrak{a}$ . Then, if  $D_{\beta,j} := X_{\beta,j}X_{-\beta,j} + X_{-\beta,j}X_{\beta,j}$ , we have (cf. [GaV], p. 132)

$$\begin{aligned} (4.27) \quad \omega &= \omega_{\mathfrak{m}} + H_1^2 + \sum_{\beta \in \{\alpha, 2\alpha\}} \sum_{j=1}^{m_{\beta}} D_{\beta,j} \\ &= \omega_{\mathfrak{m}} + H_1^2 + \sum_{\beta \in \{\alpha, 2\alpha\}} m_{\beta} H_{\beta} + 2 \sum_{\beta \in \{\alpha, 2\alpha\}} \sum_{j=1}^{m_{\beta}} X_{\beta,j} X_{-\beta,j} \end{aligned}$$

where

$$(4.28) \quad \omega_{\mathfrak{m}} := - \sum_{j=1}^m Z_j^2.$$

Hence

$$P(\omega) = \omega_{\mathfrak{m}} + H_1^2 + \sum_{\beta \in \{\alpha, 2\alpha\}} m_{\beta} H_{\beta} = \omega_{\mathfrak{m}} + \frac{L^2 + 2\rho L}{B(L, L)},$$

from which we conclude

$$(4.29) \quad p_{l,s}(\omega) = p_{l,s}(\omega_{\mathfrak{m}}) + \frac{(s + \rho)^2 - 2\rho(s + \rho)}{B(L, L)} = \frac{1}{d_l} \chi_l(\omega_{\mathfrak{m}}) + \frac{s^2 - \rho^2}{8(n + 2)}.$$

To compute  $\delta_l(\omega)$  we use Formula (4.27). Observe first that if  $f \in \mathcal{D}(G; \chi_l)$  and  $Y \in \mathfrak{U}(\mathfrak{k})$ , then  $f(a_t; Y) = \frac{1}{d_l} \chi_l(Y)$ . Hence  $\delta_l(Y) = \frac{1}{d_l} \chi_l(Y)$ . In particular,

$$(4.30) \quad \delta_l(\omega_m) = \frac{1}{d_l} \chi_l(\omega_m).$$

Write

$$(4.31) \quad X_{\beta,j} = Y_{\beta,j} + P_{\beta,j} \quad \text{with} \quad Y_{\beta,j} \in \mathfrak{k}, P_{\beta,j} \in \mathfrak{p}.$$

A standard computation (cf. e.g. [W2], p.278) then gives for  $f \in \mathcal{D}(G; \chi_l)$  and  $t > 0$

$$f(a_t; D_{\beta,j}) = \coth(t\beta(L))f(a_t; H_\beta) + \frac{4}{d_l} \frac{1 - \cosh(t\beta(L))}{\sinh^2(t\beta(L))} \chi_l(Y_{\beta,j}^2) f(a_t)$$

i.e.

$$(4.32) \quad \delta_l(D_{\beta,j}) = \coth(t\beta(L))H_\beta + \frac{4}{d_l} \frac{1 - \cosh(t\beta(L))}{\sinh^2(t\beta(L))} \chi_l(Y_{\beta,j}^2).$$

Notice that  $\chi_l(Y_{\alpha,j}^2) = 0$  for all  $j = 1, \dots, m_\alpha$ .

For  $h = i, j, k$ , let  $Y_h$  denote the tangent vector at  $e$  to the 1-parameter subgroup  $t \mapsto \cos t + h \sin t$  in  $\text{Sp}(1)$ . Explicit choices of the orthonormal bases in  $\mathfrak{m}$  and  $\mathfrak{g}_{2\alpha}$  prove that

$$(4.33) \quad \chi_l(\omega_m) = -2 \sum_{j=1}^3 \chi_l(Y_{2\alpha,j}^2) = -\frac{1}{8(n+2)} \sum_{h \in \{i,j,k\}} \text{tr} [\tau_l(Y_h)^2].$$

As shown in [T1], p.381, there exists an orthonormal basis  $\{v_p\}_{p=-l}^l$  in  $V_l$  such that

$$\begin{aligned} \tau_l(Y_i)v_p &= -2ipv_p \\ \tau_l(Y_j)v_p &= -i\alpha_{p+1}^l v_{p+1} \\ \tau_l(Y_k)v_p &= -\alpha_{p+1}^l v_{p+1} + \alpha_p^l v_{p-1} \end{aligned}$$

where

$$\alpha_p^l := [(l+p)(l-p+1)]^{1/2}.$$

It follows that for  $h = i, j, k$

$$(4.34) \quad \text{tr} [\tau_l(Y_h)^2] = -\frac{4}{3} l(l+1)(2l+1).$$

Identify  $A$  with  $\mathbf{R}$  and  $L$  with  $\frac{d}{dt}$  under the isomorphism  $t \mapsto \exp(tL) = a_t$ . Formulas (4.27), (4.30) and (4.32)–(4.34) then prove the following proposition.

4.3. PROPOSITION. *Let  $\tau_l$  be an irreducible unitary representation of  $K_1$  of dimension  $2l + 1$ . Then*

1. *The  $\tau_l$ -radial component of the Casimir operator  $\omega$  is*

$$\delta_l(\omega) = \frac{1}{8(n+2)} \left\{ \frac{d^2}{dt^2} + [(4n-1) \coth t + 3 \tanh t] \frac{d}{dt} + \frac{4l(l+1)}{\cosh^2 t} + 4l(l+1) \right\}.$$

2. *For every  $s \in \mathbf{C}$*

$$(4.35) \quad p_{l,s}(\omega) = \frac{1}{8(n+2)} [4l(l+1) + s^2 - \rho^2].$$

3. *For every  $s \in \mathbf{C}$ , the function  $\zeta_{l,s}(t) := \zeta_{l,s}(a_t)$  satisfies the differential equation on  $(0, +\infty)$*

$$(4.36) \quad u'' + [(4n-1) \coth t + 3 \tanh t] u' + \frac{4l(l+1)}{\cosh^2 t} u = (s^2 - \rho^2) u.$$

The substitution  $v(t) = (\cosh t)^{-2l} u(t)$  transforms the differential equation (4.36) into the Jacobi differential equation (cf. [K2], p. 6)

$$(4.37) \quad v'' + [(4n-1) \coth t + (4l+3) \tanh t] v' = (s^2 - \tilde{\rho}^2) v$$

with parameters  $\alpha = 2n - 1$  and  $\beta = 2l + 1$ . Here  $\tilde{\rho} := \alpha + \beta + 1 = \rho + 2l$ . The Jacobi function

$$(4.38) \quad \begin{aligned} \phi_{is}^{(2n-1, 2l+1)}(t) &:= F \left( \frac{\tilde{\rho} + s}{2}, \frac{\tilde{\rho} - s}{2}; 2n; -\sinh^2 t \right) \\ &= F \left( \frac{\rho + s}{2} + l, \frac{\rho - s}{2} + l; 2n; -\sinh^2 t \right) \end{aligned}$$

is the unique solution  $v$  to (4.37) satisfying  $v(0) = 1$ ,  $v'(0) = 0$ . (In (4.38),  $F(a, b; c; z)$  denotes the analytic branch on  $\mathbf{C} \setminus [1, \infty)$  of the hypergeometric function.)

The  $\tau_l$ -spherical function  $\zeta_{l,s}$  is therefore explicitly given by

$$(4.39) \quad \begin{aligned} \zeta_{l,s}(t) := \zeta_{l,s}(a_t) &= (\cosh t)^{2l} \phi_{is}^{(2n-1, 2l+1)}(t) \\ &= (\cosh t)^{2l} F \left( \frac{\rho + s}{2} + l, \frac{\rho - s}{2} + l; 2n; -\sinh^2 t \right). \end{aligned}$$

Formula (4.39) has been previously determined by Takahaski ([T2], Formula (7), p. 225) by direct integration of (3.23), using the following expression of  $\chi_l$  in terms of Gegenbauer polynomials:

$$(4.40) \quad \chi_l(k_1) = C_{2l}^1(\Re k_1) = \frac{\sin((2l+1)\vartheta)}{\sin \vartheta} \quad \text{if } \Re k_1 = \cos \vartheta.$$

Formula (4.35) shows that  $p_{l,s}(\omega)$  is an even function of  $s$  which assumes arbitrary complex values as  $s$  varies in  $\mathbf{C}$ . The following corollary can therefore be deduced from Theorem 3.1 and Proposition 4.3.

4.4. COROLLARY. *The  $\tau_l$ -spherical functions are exactly the functions  $\{\zeta_{l,s} : s \in \mathbf{C}\}$  given by Formulas (3.24) and (4.39). Further,  $\zeta_{l,s}$  satisfies  $\zeta_{l,s}(g) = \zeta_{l,s}(g^{-1})$  for all  $g \in G$ . Moreover,  $\zeta_{l,s} = \zeta_{l,s'}$  if and only if  $s = \pm s'$ .*

The functional equation (3.15) with  $g_1 = a_t$  and  $g_2 = a_\tau$  becomes (cf. [T2], Théorème 1, p.227)

$$(4.41) \quad \zeta_{l,s}(t)\zeta_{l,s}(\tau) = \int_0^\infty K_l(t, \tau, u)\zeta_{l,s}(u)\Delta(u) du$$

where  $\Delta$  is as in (1.7) and the kernel  $K_l(t, \tau, u)$  is defined as follows. Set

$$B := \frac{\cosh^2 t + \cosh^2 \tau + \cosh^2 u - 1}{2 \cosh t \cosh \tau \cosh u}.$$

Then

$$(4.42) \quad K_l(t, \tau, u) := \frac{2^{-2\rho}\Gamma(2n)}{\sqrt{\pi}\Gamma(2n - \frac{1}{2})} \frac{(\cosh t \cosh \tau \cosh u)^{2n-3}}{(\sinh t \sinh \tau \sinh u)^{4n-2}} (1 - B^2)^{2n-\frac{3}{2}} \\ \times F\left(2n + 2l, 2n - 2l - 2; 2n - \frac{1}{2}; \frac{1}{2}(1 - B)\right)$$

if  $B < 1$ , and  $K_l(t, \tau, u) := 0$  if  $B \geq 1$ . Using (4.39) and Formula (7.11) in [K2], one can prove that (4.41) holds also outside our group-theoretical setting for all  $l \in \mathbf{R}$  satisfying  $2n - 1 > 2l \geq 0$ .

## 5. THE POSITIVE DEFINITE $\tau_l$ -SPHERICAL FUNCTIONS

A continuous function  $\zeta$  on a locally compact group  $G$  is said to be *positive definite* if for every  $f \in C_c(G)$

$$\int_G \int_G \zeta(x^{-1}y)f(x)\overline{f(y)} dx dy \geq 0.$$

In this section we establish which among the  $\zeta_{l,s}$  are positive definite.

Let us first introduce some notation and recall some definitions. Let  $G$  be a semisimple Lie group with finite center, and let  $K$  be a maximal compact subgroup of  $G$ .  $\mathfrak{g}$  and  $\mathfrak{k}$  ( $\subset \mathfrak{g}$ ) are the Lie algebras of  $G$  and  $K$ , respectively. A (strongly continuous) representation  $T$  of  $G$  on a Banach space  $\mathcal{H}$  is denoted by  $(T, \mathcal{H})$ . We may simply speak of the representation  $T$  if  $\mathcal{H}$  is understood. Irreducibility for  $T$  always means topological irreducibility (= no closed proper invariant subspaces). Let  $\widehat{K}$  denote the set of equivalence classes