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2.2 A Fubini counterexample

In this section, we give an example of (2). Let $\mathcal{A}$ be as in (5) and define $\mu|_{\mathcal{A}}$ by

$$\mu(A) := \begin{cases} 0 & \text{if } A \text{ is meager}, \\ 1 & \text{if } A \text{ is comeager}. \end{cases} \tag{11}$$

This is possible, since no set $A \subset \mathbf{R}$ is simultaneously meager and comeager, for otherwise $\varnothing = A \cap A^c$ would be comeager, in contradiction to Baire's theorem. It is easy to check that μ is a probability measure on $(\mathbf{R}, \mathcal{A})$. Let again $\nu := \lambda :=$ Lebesgue measure on $\mathcal{B} := \mathcal{B}(\mathbf{R})$, and choose $A \in \mathcal{A}$ meager with $\lambda(A^c) = 0$. Then $1_A(\cdot + y)$ is $\mathcal{A}$-measurable with

$$\int_{\mathbf{R}} 1_A(x+y)\, d\mu(x) = \mu(A - y) = 0 \qquad (y \in \mathbf{R}).$$

On the other hand, we have

$$\int_{\mathbf{R}} 1_A(x+y)\, d\nu(y) = \lambda(A - x) = \infty \qquad (x \in \mathbf{R}).$$

Hence (2) is obviously true in this case.

3. Measurability

Here is a positive result, having a certain measurability property of F from (1) among its conclusions. An application of this occurs in Mattner (1999).

3.1. Theorem. *Let $(\mathcal{X}, \mathcal{A}, \mu)$ and $(\mathcal{Y}, \mathcal{B}, \nu)$ be σ-finite measure spaces, let $f\colon \mathcal{X} \times \mathcal{Y} \to [0, \infty]$ be a function measurable with respect to the product σ-algebra $\mathcal{A} \otimes \mathcal{B}$, and put*

$$\begin{aligned}
\mathcal{A}_0 &:= \sigma(\{f(\cdot, y) : y \in \mathcal{Y}\}), \\
\mathcal{B}_0 &:= \sigma(\{f(x, \cdot) : x \in \mathcal{X}\}), \\
\overline{\mathcal{A}}_0 &:= \{A \in \mathcal{A} : \exists A_0 \in \mathcal{A}_0 \text{ with } A = A_0 \quad [\mu]\}, \\
\overline{\mathcal{B}}_0 &:= \{B \in \mathcal{B} : \exists B_0 \in \mathcal{A}_0 \text{ with } B = B_0 \quad [\nu]\}, \\
\overline{\mathcal{A}_0 \otimes \mathcal{B}_0} &:= \{C \in \mathcal{A} \otimes \mathcal{B} : \exists C_0 \in \mathcal{A}_0 \otimes \mathcal{B}_0 \text{ with } C = C_0 \quad [\mu \otimes \nu]\}.
\end{aligned}$$

Then f is $\overline{\mathcal{A}_0 \otimes \mathcal{B}_0}$-measurable, $\int_{\mathcal{Y}} f(\cdot, y)\, d\nu(y)$ is $\overline{\mathcal{A}}_0$-measurable, and $\int_{\mathcal{X}} f(x, \cdot)\, d\mu(x)$ is $\overline{\mathcal{B}}_0$-measurable.