**Zeitschrift:** L'Enseignement Mathématique

**Herausgeber:** Commission Internationale de l'Enseignement Mathématique

**Band:** 45 (1999)

**Heft:** 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** PRODUCT MEASURABILITY, PARAMETER INTEGRALS, AND A FUBINI COUNTEREXAMPLE

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**Bibliographie**

**DOI:** https://doi.org/10.5169/seals-64449

$$\int_{B_0} f(\cdot,y)\,d\overline{\nu}_0(y) = \int_{B_0} \widetilde{f}(\cdot,y)\,d\overline{\nu}_0(y) \qquad [\overline{\mu}_0]\,.$$

Trivially, this remains true if $[\overline{\mu}_0]$ is replaced by $[\mu]$, and an integration yields

$$\int\limits_A \int\limits_{B_0} f(x,y)\,d\overline{\nu}_0(y)\,d\mu(x) = \int\limits_A \int\limits_{B_0} \widetilde{f}(x,y) d\overline{\nu}_0(y)\,d\mu(x) \tag{18}$$

$(A \in \mathcal{A},\ B_0 \in \overline{\mathcal{B}}_0)$. We now want to interchange the order of integrations. Since $\widetilde{f}$ is trivially $\mathcal{A} \otimes \overline{\mathcal{B}}_0$-measurable, we may obviously do this on the right hand side of (18). To do the same on the left hand side, we rewrite it successively as

$$\int_A \int_{B_0} f(x,y)\,d\nu(y)\,d\mu(x) = \int_{B_0} \int_A f(x,y)\,d\mu(x)\,d\nu(y) = \int_{B_0} \int_A f(x,y)\,d\mu(x)\,d\overline{\nu}_0(y)\,,$$

where the last equality follows from a second application of Claim 1, with the role of the variables interchanged. Thus (18) yields

$$\int\limits_{B_0} \int\limits_A f(x,y)\,d\mu(x)\,d\overline{\nu}_0(y) = \int\limits_{B_0} \int\limits_A \widetilde{f}(x,y) d\mu(x)\,d\overline{\nu}_0(y) \tag{19}$$

$(A \in \mathcal{A},\ B_0 \in \overline{\mathcal{B}}_0)$. Now the argument leading from (17) to (18) can be repeated to lead from (19) to a corresponding statement with $B$ in place of $B_0$, $\nu$ in place of $\overline{\nu}_0$, and $\mathcal{B}$ in place of $\overline{\mathcal{B}}_0$, which is equivalent to

$$\int_{A\times B} f\,d\mu \otimes \nu = \int_{A\times B} \widetilde{f}\,d\mu \otimes \nu \qquad (A \in \mathcal{A},\ B \in \mathcal{B})\,.$$

This shows that $f = \widetilde{f}\ \ [\mu \otimes \nu]$, which yields the desired conclusion. □

ACKNOWLEDGEMENT. The present work was supported by a Heisenberg grant of the Deutsche Forschungsgemeinschaft. I thank H. von Weizsäcker for helpful remarks on an earlier version.

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*(Reçu le 2 mars 1999)*

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