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THEOREM 1. *There is a morphism of algebraic varieties  $\theta: X - S \rightarrow J_m$  satisfying the following properties:*

- (a) *The extension of  $\theta$  to the group of divisors on  $X$  prime to  $S$  induces, by passing to quotient, an isomorphism between the group  $C_m^0$  of classes of divisors of degree zero with respect to  $m$ -equivalence and the group  $J_m$ .*
- (b) *The extension of  $\theta$  to  $(X - S)^{(\pi)}$  induces a birational map from  $X^{(\pi)}$  to  $J_m$ .*

The following theorem characterizes  $J_m$  by a universal property:

THEOREM 2. *Let  $f: X \rightarrow G$  be a rational map from  $X$  to a commutative algebraic group  $G$  and assume  $m$  is a modulus for  $f$ . Then there is a unique homomorphism  $F: J_m \rightarrow G$  of algebraic groups such that  $f = F \circ \theta + f(P_0)$ .*

*Proof.* Replacing  $f$  by  $f - f(P_0)$ , we may assume  $f(P_0) = 0$ . Since  $m$  is a modulus for  $f$ , the extension of  $f$  to the group of divisors of  $X$  prime to  $S$  induces a homomorphism  $C_m^0 \rightarrow G$  by passing to quotient. By Theorem 1 (a) we have  $J_m \cong C_m^0$  as groups. So we have a homomorphism of groups  $F: J_m \rightarrow G$  such that  $f = F\theta$ . It remains to prove  $F$  is a morphism of algebraic varieties. By Theorem 1 (b) we have a birational map  $\theta: (X - S)^{(\pi)} \rightarrow J_m$ . Denote the extension of  $f$  to  $(X - S)^{(\pi)}$  by  $f'$ . Then  $F\theta = f'$ . Since  $\theta$  is birational, it induces an isomorphism between an open subvariety of  $(X - S)^{(\pi)}$  and an open subvariety of  $J_m$ . Moreover  $f'$  is a morphism of algebraic varieties. Hence  $F$  is a morphism of algebraic varieties when restricted to some open subset of  $J_m$ . The whole  $J_m$  can be obtained from this open subset by translation. So  $F$  is a morphism of algebraic varieties.

## 6. GENERALIZED JACOBIANS AND PICARD SCHEMES

In this section we prove  $J_m$  is the Picard scheme of  $X_m$ .

Let  $T$  be a  $k$ -scheme. Consider the Cartesian square

$$\begin{array}{ccc} X_m \times T & \longrightarrow & X_m \\ q \downarrow & & \downarrow \\ T & \longrightarrow & \text{spec}(k) . \end{array}$$

We have  $q_*\mathcal{O}_{X_m \times T} = \mathcal{O}_T$  by [EGA] III, § 1.4.15, the fact  $H^0(X_m, \mathcal{O}_{X_m}) = k$ , and the fact that  $T \rightarrow \text{spec}(k)$  is flat. The morphism  $q$  has a section  $s: T \rightarrow X_m \times T$ ,  $t \mapsto (P_0, t)$ .

LEMMA 6.1. *Let  $\mathcal{L}_1$  and  $\mathcal{L}_2$  be two invertible sheaves on  $X_m \times T$ . Assume  $\mathcal{L}_1 \cong \mathcal{L}_2$ . Then the canonical map  $\text{Hom}(\mathcal{L}_1, \mathcal{L}_2) \rightarrow \text{Hom}(s^*\mathcal{L}_1, s^*\mathcal{L}_2)$  induced by  $s$  is bijective.*

*Proof.* Since  $\mathcal{L}_1 \cong \mathcal{L}_2$ , it is enough to show that the canonical map  $\text{Hom}(\mathcal{L}_1, \mathcal{L}_1) \rightarrow \text{Hom}(s^*\mathcal{L}_1, s^*\mathcal{L}_1)$  is bijective. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{X_m \times T}(X_m \times T) & \longrightarrow & \mathcal{O}_T(T) \\ \downarrow & & \downarrow \\ \text{Hom}(\mathcal{L}_1, \mathcal{L}_1) & \longrightarrow & \text{Hom}(s^*\mathcal{L}_1, s^*\mathcal{L}_1), \end{array}$$

where the horizontal arrows are induced by  $s$ . We have

$$\begin{aligned} \text{Hom}(\mathcal{L}_1, \mathcal{L}_1) &\cong \text{Hom}(\mathcal{O}_{X_m \times T}, \mathcal{L}_1 \otimes \mathcal{L}_1^{-1}) \\ &\cong \text{Hom}(\mathcal{O}_{X_m \times T}, \mathcal{O}_{X_m \times T}) \cong \mathcal{O}_{X_m \times T}(X_m \times T). \end{aligned}$$

Hence the left vertical arrow in the above diagram is bijective. Similarly the right vertical arrow is also bijective. Since  $q_*\mathcal{O}_{X_m \times T} = \mathcal{O}_T$ , we have  $\mathcal{O}_{X_m \times T}(X_m \times T) \cong \mathcal{O}(T)$ , and the upper horizontal arrow is bijective. Hence  $\text{Hom}(\mathcal{L}_1, \mathcal{L}_1) \cong \text{Hom}(s^*\mathcal{L}_1, s^*\mathcal{L}_1)$  by the commutativity of the above diagram.

LEMMA 6.2. *Let  $\{U_i\}$  be an open covering of  $T$  and let  $\mathcal{L}_i$  be invertible sheaves on  $X_m \times U_i$ . Assume  $s^*\mathcal{L}_i \cong \mathcal{O}_{U_i}$  and  $\mathcal{L}_i|_{X_m \times (U_i \cap U_j)} \cong \mathcal{L}_j|_{X_m \times (U_i \cap U_j)}$ . Then there exists an invertible sheaf  $\mathcal{L}$  on  $X_m \times T$  such that  $\mathcal{L}|_{X_m \times U_i} \cong \mathcal{L}_i$  and  $s^*\mathcal{L} \cong \mathcal{O}_T$ . Moreover  $\mathcal{L}$  is unique up to isomorphism.*

*Proof.* Fix an isomorphism  $\alpha_i: s^*\mathcal{L}_i \rightarrow \mathcal{O}_{U_i}$  for each  $i$ . Let

$$\alpha_{ij}: s^*\mathcal{L}_i|_{U_i \cap U_j} \rightarrow s^*\mathcal{L}_j|_{U_i \cap U_j}$$

be the isomorphism  $(\alpha_j|_{U_i \cap U_j})^{-1} \circ (\alpha_i|_{U_i \cap U_j})$ . By Lemma 6.1 the canonical map

$$\text{Hom}(\mathcal{L}_i|_{X_m \times (U_i \cap U_j)}, \mathcal{L}_j|_{X_m \times (U_i \cap U_j)}) \rightarrow \text{Hom}(s^*\mathcal{L}_i|_{U_i \cap U_j}, s^*\mathcal{L}_j|_{U_i \cap U_j})$$

is bijective. So  $\alpha_{ij}$  can be lifted uniquely to an isomorphism

$$A_{ij}: \mathcal{L}_i|_{X_m \times (U_i \cap U_j)} \rightarrow \mathcal{L}_j|_{X_m \times (U_i \cap U_j)}.$$

By the uniqueness of the lifting and the fact that  $\alpha_{jk}\alpha_{ij} = \alpha_{ik}$  on  $U_i \cap U_j \cap U_k$ , we have  $A_{jk}A_{ij} = A_{ik}$  on  $X_m \times (U_i \cap U_j \cap U_k)$ . So  $A_{ij}$  defines glueing data and we can glue the  $\mathcal{L}_i$  together to get an invertible sheaf  $\mathcal{L}$  on  $X_m \times T$ . By the construction of  $\mathcal{L}$  we have  $s^*\mathcal{L} \cong \mathcal{O}_T$ . This proves the existence of  $\mathcal{L}$ . Similarly using Lemma 6.1 one can prove  $\mathcal{L}$  is unique up to isomorphism.

LEMMA 6.3. Assume  $T$  is integral. Let  $\mathcal{L}_1$  and  $\mathcal{L}_2$  be two invertible sheaves on  $X_m \times T$  satisfying  $\mathcal{L}_{1,t} \cong \mathcal{L}_{2,t}$  for all  $t \in T$ . Then there is an invertible sheaf  $\mathcal{M}$  on  $T$  such that  $\mathcal{L}_1 \cong \mathcal{L}_2 \otimes q^* \mathcal{M}$ .

*Proof.* Let  $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ . Then  $\mathcal{L}_t \cong \mathcal{O}_{X_m}$ . It suffices to show that  $\mathcal{L} \cong q^* \mathcal{M}$  for some invertible sheaf  $\mathcal{M}$  on  $T$ . We have  $H^0(X_m, \mathcal{L}_t) = H^0(X_m, \mathcal{O}_{X_m}) = k$ . By Theorem 1.1(c), the sheaf  $q_* \mathcal{L}$  is invertible and  $q_* \mathcal{L} \otimes k(t) = H^0(X_m, \mathcal{L}_t)$ . So the restriction  $(q^* q_* \mathcal{L})_t \rightarrow \mathcal{L}_t$  of the canonical map  $q^* q_* \mathcal{L} \rightarrow \mathcal{L}$  to the fiber of  $q$  at  $t \in T$  is  $H^0(X_m, \mathcal{L}_t) \otimes \mathcal{O}_{X_m} \rightarrow \mathcal{L}_t$ , which is an isomorphism since  $\mathcal{L}_t \cong \mathcal{O}_{X_m}$ . By Nakayama's Lemma, the canonical map  $q^* q_* \mathcal{L} \rightarrow \mathcal{L}$  is surjective. But since it is a homomorphism of invertible sheaves, it must be bijective. Hence  $\mathcal{L} \cong q^* q_* \mathcal{L}$ .

Now we use the above lemmas to construct a canonical invertible sheaf on  $X_m \times J_m$ .

On  $X_m \times (X - S)^{(\pi)}$  we have the invertible sheaf corresponding to the divisor  $\mathcal{D} - p^*(\pi P_0)$ , where  $\mathcal{D}$  is the universal relative effective Cartier divisor and  $p: X_m \times (X - S)^{(\pi)} \rightarrow X_m$  is the projection. Since  $\theta: (X - S)^{(\pi)} \rightarrow J_m$  is birational, there exist open subsets  $U$  in  $(X - S)^{(\pi)}$  and  $V$  in  $J_m$  such that  $\theta$  induces an isomorphism  $U \cong V$ . Hence we can push-forward the above invertible sheaf on  $X_m \times (X - S)^{(\pi)}$  to get an invertible sheaf  $\mathcal{L}_V$  on  $X_m \times V$ . For each  $t \in J_m$ , denote by  $\mathcal{L}(t)$  the invertible sheaf on  $X_m$  corresponding to the divisor class in  $C_m^0$  that is mapped to  $t \in J_m$  under the canonical isomorphism  $C_m^0 \cong J_m$ . Obviously the restriction  $\mathcal{L}_{V,t}$  of  $\mathcal{L}_V$  to the fiber of the projection  $q: X_m \times J_m \rightarrow J_m$  at  $t \in V$  is isomorphic to  $\mathcal{L}(t)$ . The invertible sheaf  $\mathcal{L}_V \otimes (q^* s^* \mathcal{L}_V)^{-1}$  has the same property, where  $s: J_m \rightarrow X_m \times J_m$  is the section  $t \mapsto (P_0, t)$ . Thus replacing  $\mathcal{L}_V$  by  $\mathcal{L}_V \otimes (q^* s^* \mathcal{L}_V)^{-1}$  if necessary, we may assume that  $s^* \mathcal{L}_V \cong \mathcal{O}_V$ .

For each  $a \in J_m$ , let  $T_{-a}: J_m \rightarrow J_m$  be the translation  $t \mapsto t - a$ . Consider the invertible sheaf  $\mathcal{L}_{a+V} = (\text{id} \times T_{-a})^* \mathcal{L}_V \otimes p^* \mathcal{L}(a)$  on  $X_m \otimes (a + V)$ , where  $p: X_m \times J_m \rightarrow X_m$  is the projection. The restriction  $\mathcal{L}_{a+V,a+t}$  of  $\mathcal{L}_{a+V}$  to the fiber of  $q$  at  $a + t \in a + V$  is

$$((\text{id} \times T_{-a})^* \mathcal{L}_V \otimes p^* \mathcal{L}(a))_{a+t} = \mathcal{L}_{V,t} \otimes \mathcal{L}(a) = \mathcal{L}(t) \otimes \mathcal{L}(a) = \mathcal{L}(a + t),$$

that is,  $\mathcal{L}_{a+V,a+t} = \mathcal{L}(a + t)$ . Hence for any  $t \in V \cap (a + V)$ , we have  $\mathcal{L}_{V,t} = \mathcal{L}_{a+V,t}$ . By Lemma 6.3, we have

$$\mathcal{L}_V|_{X_m \times (V \cap (a + V))} \cong \mathcal{L}_{a+V}|_{X_m \times (V \cap (a + V))} \otimes q^* \mathcal{M}$$

for some invertible sheaf  $\mathcal{M}$  on  $V \cap (a + V)$ . But since  $s^* \mathcal{L}_V \cong \mathcal{O}_V$ , we also have  $s^* \mathcal{L}_{a+V} = \mathcal{O}_{a+V}$ . Hence  $\mathcal{M} \cong \mathcal{O}_{V \cap (a + V)}$ . Therefore  $\mathcal{L}_V|_{X_m \times (V \cap (a + V))} \cong$

$\mathcal{L}_{a+V}|_{X_m \times (V \cap (a+V))}$ . By Lemma 6.2, we can glue  $\mathcal{L}_{a+V}$  ( $a \in J_m$ ) together to get an invertible sheaf  $\mathcal{L}_{J_m}$  on  $X_m \times J_m$ . It has the property that its restriction to the fiber of  $q$  at  $t \in J_m$  is isomorphic to  $\mathcal{L}(t)$  and  $s^* \mathcal{L}_{J_m} \cong \mathcal{O}_{J_m}$ .

Define

$$P^0(T) = \{\mathcal{L} \in \text{Pic}(X_m \times T) \mid \deg(\mathcal{L}) = 0\} / q^* \text{Pic}(T),$$

where  $\deg(\mathcal{L})$  is defined as the leading coefficient of  $\chi(\mathcal{L}_t^{\otimes n})$  as a polynomial in  $n$ . Since  $s^* q^* = \text{id}$ , we may define

$$P^0(T) = \{\mathcal{L} \in \text{Pic}(X_m \times T) \mid \deg(\mathcal{L}) = 0 \text{ and } s^* \mathcal{L} \cong \mathcal{O}_T\}$$

as well. In particular, we have  $\mathcal{L}_{J_m} \in P^0(J_m)$ . Using the first definition of  $P^0(T)$  and Lemma 6.3, one can show that the pull-back of  $\mathcal{L}_{J_m}$  by  $\text{id} \times \theta: X_m \times (X - S)^{(\pi)} \rightarrow X_m \times J_m$  is the invertible sheaf on  $X_m \times (X - S)^{(\pi)}$  corresponding to the divisor  $\mathcal{D} - p^*(\pi P_0)$ .

The following theorem says that  $J_m$  is the Picard scheme of  $X_m$ .

**THEOREM 3.** *The functor  $T \rightarrow P^0(T)$  is represented by  $J_m$ . More precisely, for any invertible sheaf  $\mathcal{L}$  on  $X_m \times T$  of degree 0 satisfying  $s^* \mathcal{L} \cong \mathcal{O}_T$ , there is one and only one morphism of schemes  $f: T \rightarrow J_m$  such that  $\mathcal{L}$  is the pull-back of  $\mathcal{L}_{J_m}$  by  $\text{id} \times f: X_m \times T \rightarrow X_m \times J_m$ .*

*Proof.* Let  $V_0 = \{D \in (X - S)^{(\pi)} \mid l_m(D) = 1, l(D - m) = 0\}$ . By Lemma 3.3, we know  $V_0$  is non-empty and open in  $(X - S)^{(\pi)}$ . Note that for every  $D \in V_0$ , there is one and only one effective divisor in  $X_m$  that is  $m$ -equivalent to  $D$ . Hence the restriction  $\theta|_{V_0}$  of  $\theta: (X - S)^{(\pi)} \rightarrow J_m$  to  $V_0$  is injective. By [EGA] III, §4.4.9,  $\theta|_{V_0}$  is an open immersion.

Consider the Cartesian square

$$\begin{array}{ccc} X_m \times T & \xrightarrow{p} & X_m \\ q \downarrow & & \downarrow \\ T & \longrightarrow & \text{spec}(k). \end{array}$$

Let  $\mathcal{L}' = \mathcal{L} \otimes p^* \mathcal{L}(\pi P_0)$ , where  $\mathcal{L}(\pi P_0)$  is the invertible sheaf on  $X_m$  corresponding to the divisor  $\pi P_0$ . Let us prove the theorem under the extra assumption that for every  $t \in T$ , we have  $\dim H^0(X_m, \mathcal{L}'_t) = 1$  and  $\dim H^0(X, \mathcal{L}'_t \otimes \mathcal{L}(-m)) = 0$ , where  $\mathcal{L}(-m)$  is the invertible sheaf on  $X$  corresponding to the divisor  $-m$ . By the Riemann-Roch theorem, for every  $t \in T$ , we have  $\dim H^1(X_m, \mathcal{L}'_t) = 0$ . By Theorem 1.1 (d) the sheaf  $q_* \mathcal{L}'$  is invertible. The canonical map  $q^* q_* \mathcal{L}' \rightarrow \mathcal{L}'$  induces

$$s: \mathcal{O}_{X_m \times T} \rightarrow \mathcal{L}' \otimes (q^* q_* \mathcal{L}')^{-1}.$$

Using Remark 2.1, one can show that the pair  $(\mathcal{L}' \otimes (q^* q_* \mathcal{L}')^{-1}, s)$  defines a relative effective Cartier divisor on  $(X_m \times T)/T$ . By Proposition 3.1, there exists a unique morphism of schemes  $g: T \rightarrow (X - S)^{(\pi)}$  such that the pull-back by  $\text{id} \times g$  of the universal relative effective Cartier divisor  $\mathcal{D}$  is the divisor defined by  $(\mathcal{L}' \otimes (q^* q_* \mathcal{L}')^{-1}, s)$ . Let  $f = \theta g$ . Then the pull-back of  $\mathcal{L}_{J_m}$  by  $\text{id} \times f$  is  $\mathcal{L}$ . This proves the existence of  $f$ . To prove  $f$  is unique, assume  $f: T \rightarrow J_m$  is a morphism such that the pull-back of  $\mathcal{L}_{J_m}$  by  $\text{id} \times f$  is  $\mathcal{L}$ . By our extra assumption, we must have  $\text{Im}(f) \subset \theta(V_0)$ . But  $\theta|_{V_0}$  is an open immersion. So there exists a morphism  $g: T \rightarrow (X - S)^{(\pi)}$  such that  $f = \theta g$ . We leave it to the reader to prove that the pull-back of the universal relative effective Cartier divisor  $\mathcal{D}$  by  $\text{id} \times g$  is the divisor defined by the pair  $(\mathcal{L}' \otimes (q^* q_* \mathcal{L}')^{-1}, s)$ . By Proposition 3.1, such kind of  $g$  is unique. So  $f$  is also unique.

Now let us prove the theorem. Let  $t_0$  be a point in  $T$ . For every point  $D \in (X - S)^{(\pi)}$ , denote by  $\mathcal{L}(D)$  the invertible sheaf on  $X$  or on  $X_m$  corresponding to the divisor  $D$ . By Lemma 3.3, the set

$$\{D \in (X - S)^{(\pi)} \mid \dim H^0(X_m, \mathcal{L}_{t_0} \otimes \mathcal{L}(D)) = 1, \dim H^0(X, \mathcal{L}_{t_0} \otimes \mathcal{L}(D - m)) = 0\}$$

is non-empty (and open). Fix an element  $D$  in this set. Consider the set

$$U_{t_0} = \{t \in T \mid \dim H^0(X_m, \mathcal{L}_t \otimes \mathcal{L}(D)) = 1, \dim H^0(X, \mathcal{L}_t \otimes \mathcal{L}(D - m)) = 0\}.$$

This set is open by the Riemann-Roch theorem and Theorem 1.1 (b). Obviously it contains  $t_0$ . So  $U_{t_0}$  is an open neighbourhood of  $t_0$ . By the theorem with the extra assumption that we have already proved, there exists a unique morphism  $f'_{U_{t_0}}: U_{t_0} \rightarrow J_m$  such that the pull-back of  $\mathcal{L}_{J_m}$  by  $\text{id} \times f'_{U_{t_0}}$  is  $(\mathcal{L} \otimes p^* \mathcal{L}(D - \pi P_0))|_{X_m \times U_{t_0}}$ . Put  $f_{U_{t_0}} = f'_{U_{t_0}} + a$ , where  $a$  is the point in  $J_m$  corresponding to the divisor class  $\pi P_0 - D$  in  $C_m^0$ . Obviously the pull-back of  $\mathcal{L}_{J_m}$  by the morphism  $\text{id} \times f_{U_{t_0}}$  is  $\mathcal{L}|_{X_m \times U_{t_0}}$ . Moreover, such an  $f_{U_{t_0}}$  is unique. So we can glue  $f_{U_{t_0}}$  together to get  $f: T \rightarrow J_m$ .