Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 45 (1999)

Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: GENERALIZED FØLNER CONDITION AND THE NORMS OF RANDOM WALK OPERATORS ON GROUPS

Autor: UK, Andrzej

Kapitel: 3. Remarks

DOI: https://doi.org/10.5169/seals-64454

This implies that

$$\frac{1}{\#S}|A|_{f^2} = \sum_{\gamma\in A} f^2(\gamma)\,,$$

$$\frac{1}{\lambda(\#S)^2}|\partial A|_{f^2} \leq \sum_{\gamma\in\partial A} f^2(\gamma) \leq \lambda|\partial A|_{f^2}\,.$$

By Theorem 2, the first condition implies the second one. □

REMARK. The proof of Theorem 3 can easily be generalized to the case where P is a convolution operator with a finitely supported probability measure.

3. REMARKS

We will now make some comments about Theorems 2 and 3. We will state some theorems about the existence of eigenfunctions for the Markov operator and discuss whether one can take in the generalized Følner condition the eigenfunctions to be in $L^2(X,\mu)$.

For simplicity we will suppose that X is a connected, locally finite graph (i.e. the degree of each vertex is finite) and we consider the *simple random walk* going with equal probability from one vertex to any of its neighbors. We associate with this random walk the simple random walk operator P defined by

$$Pf(v) = \frac{1}{N(v)}\sum_{w\sim v} f(w) \quad \text{for } f\in l^2(X,N)$$

where $N(v)$ is the degree of vertex v in X (*i.e.* the number of edges adjacent to v), where $w\sim v$ means that w and v are connected by an edge and where $l^2(X,N)$ is the space of real-valued functions f on the vertices of X such that $\sum_{x\in X} f^2(x)N(x)$ is finite.

3.1 EXISTENCE OF EIGENFUNCTIONS

THEOREM 4 ([20]). *Let X be an infinite, locally finite graph and let P be the simple random walk operator on $l^2(X,N)$. For any $\lambda \geq \|P\|$ there exists a positive eigenfunction f of P with eigenvalue λ, i.e.*

$$Pf(x) = \lambda f(x) \quad \textit{and } f(x) > 0 \textit{ for } x\in X\,.$$

For $\lambda < \|P\|$ there are no positive eigenfunctions of P with eigenvalue λ.

Proof. There are several proofs of this theorem. In [20] one can find the proof where the analogue of Perron-Frobenius theory is developed and in [11] the truncation method is used. □

3.2 EIGENFUNCTIONS IN l^2

One can ask whether the positive eigenfunctions of the random walk operator are in $l^2(X,N)$. The answer is no in the case when X is the Cayley graph of an infinite group Γ (see Theorem 5). But in the general case there are examples of eigenfunctions which are in $l^2(X,N)$ (see Proposition 2).

3.2.1 THE CASE OF GROUPS

THEOREM 5. *Let f be a positive eigenfunction of the simple random walk operator P on the group Γ generated by a finite symmetric set S, i.e. $Pf = \lambda f$. If Γ is infinite then*

$$\sum_{\gamma\in\Gamma} f^2(\gamma) = +\infty .$$

Proof. Suppose the contrary, i.e. that there is a positive eigenfunction f of the operator P for which the l^2 norm is finite:

$$Pf_0 = \lambda f_0 ,$$
$$\sum_{\gamma\in\Gamma} f_0^2(\gamma) < +\infty .$$

The second condition implies that f_0 is not constant and so there are $\gamma_0, \gamma_1 \in \Gamma$ such that

$$f_0(\gamma_0) < f_0(\gamma_1) .$$

Let us define the function f_1 as a translation of f_0 by $\gamma_0\gamma_1^{-1}$, i.e.

$$f_1(\gamma) = f_0(\gamma_0\gamma_1^{-1}\gamma) .$$

The function f_1, being the translation of f_0, is an eigenfunction of P, i.e.

$$Pf_1 = \lambda f_1 .$$

So the function $\tilde{f}$ defined as follows:

$$\tilde{f}(\gamma) = \max\{f_0(\gamma), f_1(\gamma)\} ,$$

satisfies

$$P\tilde{f} \geq \lambda \tilde{f} .$$

As f_0 and f_1 are in $l^2(\Gamma)$, the function $\widetilde{f}$ is in $l^2(\Gamma)$ as well. The functions f_0 and f_1 have the same l^2 norms and

$$f_1(\gamma_1) = f_0(\gamma_0) < f_0(\gamma_1),$$

so there exists $\gamma_2 \in \Gamma$ such that

$$f_1(\gamma_2) > f_0(\gamma_2).$$

Note that these two inequalities imply that $\widetilde{f} \geq f_0$ with equality at some points and strict inequality at some other points. Thus $g = \widetilde{f} - f_0$ satisfies $g \geq 0$, $g \neq 0$, g vanishes at some points and $Pg \geq \lambda g$. Let us prove that this implies $P\widetilde{f} \neq \lambda \widetilde{f}$. Indeed, if we had equality then $Pg = \lambda g$ as well and thus $P^n g = \lambda^n g$. Taking n large enough makes $P^n g$ non-zero at points where g vanishes, a contradiction. We have thus shown that $P\widetilde{f} \geq \lambda \widetilde{f}$ with $P\widetilde{f} \neq \lambda \widetilde{f}$.

This means that

$$\|P\widetilde{f}\|_{l^2(\Gamma)} > \lambda \|\widetilde{f}\|_{l^2(\Gamma)}.$$

Hence

$$\|P\| > \lambda.$$

But this provides the desired contradiction because by Theorem 4 there are no positive eigenfunctions of P with an eigenvalue smaller than the norm of P. □

3.2.2 The general case

It will be shown that there are examples of the infinite graph X and the simple random walk operator P for which there is a positive eigenfunction in $l^2(X, N)$. It was pointed out to us by the referee that when P is the adjacency operator, examples of infinite graphs with positive eigenvalues in l^2 can be found for instance in [5] (page 232).

Let X be a uniform tree (*i.e.* a simply connected graph) of degree 3. By a theorem of Kesten (see [9]) one knows that $\|P\| = \frac{2}{3}\sqrt{2} < 1$. Let a and b be two neighboring vertices in X. Now let X_n be a graph which is the same as the graph X, except that the edge (a, b) is subdivided into n vertices. Let I_n denote the set of vertices a, b and added vertices which we label $1, \ldots, n$ (see Figure 1). Let P_n be the simple random walk operator on X_n. One has $\|P_n\| \to_{n\to\infty} 1$. In fact we will prove:

FIGURE 1
The graph X_n

PROPOSITION 2. *For $n \geq 7$ one has*

$$\|P_n\| > \cos\left(\frac{\pi}{n+3}\right) > \frac{2\sqrt{2}}{3}.$$

For any $n_0 \geq 1$ such that $\|P_{n_0}\| > \frac{2}{3}\sqrt{2}$ the eigenfunctions of P_{n_0} corresponding to the eigenvalue $\|P_{n_0}\|$ are in $l^2(X_{n_0}, N)$.

Proof. For $n \geq 7$ let $t = \sin\left(\frac{\pi}{n+3}\right) / \sin\left(\frac{2\pi}{n+3}\right)$ so that $0 < t < 1$.

For $x \in X \setminus I_n$ let $|x|$ be the minimum of its distances from a and b. We define the function f_n on X_n as follows:

$$f_n(y) = \begin{cases} t^{|y|} & \text{for } y \in X \setminus I_n \\ \sin\left(\frac{\pi(y+1)}{n+3}\right) / \sin\left(\frac{\pi}{n+3}\right) & \text{for } y = 1, \dots, n \\ 1 & \text{for } y = a, b. \end{cases}$$

We verify that

$$P_n f_n(i) = \cos\left(\frac{\pi}{n+3}\right) f_n(i) \quad \text{for } i = 1, \dots, n$$

$$P_n f_n(x) = \frac{1}{3}\left(\cos^{-1}\left(\frac{\pi}{n+3}\right) + 2\cos\left(\frac{\pi}{n+3}\right)\right) f_n(x) \text{ for } x \in X_n \setminus \{1, \dots, n\}.$$

On the other hand for $n \geq 7$ we have $t < \frac{1}{\sqrt{3}}$ and

$$\sum_{x \in X_n \setminus I_n} f_n^2(x) N(x) = 2 \sum_{k=1}^{\infty} 2 \cdot 3^{k-1} \left(t^k\right)^2 \cdot 3 < \infty.$$

Thus f_n is in $l^2(X_n, N)$ and

$$P_n f_n \geq \cos\left(\frac{\pi}{n+3}\right) f_n.$$

So we have proved the first part of Proposition 2.

Let n_0 be such that

$$\|P_{n_0}\|_{l^2(X_{n_0},N)\to l^2(X_{n_0},N)} = \sigma > \frac{2\sqrt{2}}{3}\,.$$

Now let f be an eigenfunction of the operator P_{n_0} with the eigenvalue σ, i.e.

$$P_{n_0}f = \sigma f\,.$$

We want to show that $f \in l^2(X_{n_0}, N)$. Suppose this is not true, i.e.

$$\sum_{x\in X_{n_0}} f^2(x)N(x) = +\infty\,.$$

By Theorem 2, there exists a sequence of subsets of X_{n_0}, $A_k \subset X_{n_0}$ such that

$$\frac{\sum_{x\in\partial A_k} f^2(x)N(x)}{\sum_{x\in A_k} f^2(x)N(x)} \to_{k\to\infty} 0\,. \tag{3}$$

As I_{n_0} is a fixed finite set, the sequence $C_k = A_k \setminus I_{n_0}$ is non-empty for k sufficiently large. We need the following:

LEMMA 3. *One has*

$$\frac{\sum_{x\in\partial C_k} f^2(x)N(x)}{\sum_{x\in C_k} f^2(x)N(x)} \to_{k\to\infty} 0\,.$$

Proof. If $\sum_{x\in A_k} f^2(x)N(x) \to_{k\to\infty} \infty$ then the statement of the lemma is clear. Suppose then that for all k

$$\sum_{x\in A_k} f^2(x)N(x) \le \alpha < \infty\,. \tag{4}$$

If $A_k \cap I_{n_0} = \varnothing$ then A_k and C_k coincide. So we are interested only in those k for which $A_k \cap I_{n_0} \ne \varnothing$. Let us consider the ball B_R of radius R centered in $a \in I_{n_0}$ (*i.e.* those vertices in X_{n_0} for which at most R edges are needed to connect them to a).

Because of (3) and (4) we have that for k sufficiently large $\partial A_k \cap B_R = \varnothing$ which, by the fact that $A_k \cap I_{n_0} \ne \varnothing$, implies that $B_R \subset A_k$. But R can be chosen arbitrarily large and as f is not in $l^2(X,N)$ we get

$$\sum_{x\in A_k} f^2(x)N(x) \to_{k\to\infty} \infty\,,$$

which contradicts (4). This completes the proof of the lemma. □

On the subsets C_k the graphs X and X_{n_0} coincide. This implies:

$$\|P\|_{l^2(X,N)\to l^2(X,N)} \ge \sigma > \frac{2\sqrt{2}}{3}\,,$$

which yields the desired contradiction. This ends the proof of Proposition 2. □