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Artikel: GENERALIZED FØLNER CONDITION AND THE NORMS OF RANDOM WALK OPERATORS ON GROUPS

Autor: UK, Andrzej

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4. NORMS OF RANDOM WALK OPERATORS

Now we will show how Theorem 3 can be used in the problem of computing the norm of the random walk operator P on some groups. Our strategy is as follows: we want to find a positive eigenfunction for the operator P which satisfies the generalized Følner condition. By Theorem 3 such an eigenfunction always exists and the eigenvalue corresponding to this eigenfunction is equal to the norm of the operator P. Theorem 3 is a particular case of Theorem 2 which can also be helpful in computing the norms of more general operators as shown in Section 4.3.

4.1 FREE GROUPS

First of all, as a simple illustration of this method, we will compute the norm of the simple random walk operator on free groups, which was first done by Kesten (see [9]) using a different method.

THEOREM 6 (Kesten). *Let Γ be the free group generated by the standard symmetric set of generators S. The norm of the simple random walk operator P associated to (Γ, S) is equal to*

$$\|P\| = \frac{2\sqrt{\#S-1}}{\#S}\,.$$

Proof. The Cayley graph of (Γ, S) is a homogeneous tree T_k of degree $k = \#S$. We draw the tree T_k with level lines as in Figure 3 (level lines are marked by dotted lines). Let us choose arbitrarily a line as the line of level 0. We construct a function on vertices of this tree which depends only on the level of the vertex. For a vertex $v \in T_k$ we denote by $|v|$ its level. We define $f\colon T_k \to \mathbf{R}_+$ as follows

$$f(v) = \left(\frac{1}{\sqrt{k-1}}\right)^{|v|}.$$

One has

$$Pf = \frac{2\sqrt{k-1}}{k} f\,.$$

Let A_n be the set of vertices in T_k consisting of a chosen vertex e from the level 0 and the vertices lying below e up to the level n (in Figure 3 the vertices of A_2 are marked with circles). Then

$$\sum_{v \in A_n} f^2(v) = n + 1\,,$$

$$\sum_{v \in \partial A_n} f^2(v) = 2\,.$$

This shows that $\{A_n\}_{n=1}^{\infty}$ is a generalized Følner sequence and by Theorem 3

$$\|P\| = \frac{2\sqrt{k-1}}{k}\,. \qquad \square$$

4.1.1 Remarks on generalized growth

Let Γ be a group generated by a finite, symmetric set S. For $id \neq \gamma \in \Gamma$ we define its length $|\gamma|$ as the minimal number of generators from S needed to represent γ, i.e.

$$|\gamma| = \min\{n\,;\ \gamma = s_{i_1} \dots s_{i_n}\,,\ s_{i_j} \in S\}\,,$$

and we declare $|id| = 0$.

The growth function (see [10], [18]) of the pair (Γ, S) associates to each integer $n \geq 0$ the number $\beta(\Gamma, S)(n)$ of elements $\gamma \in \Gamma$ such that $|\gamma| \leq n$, i.e.

$$\beta(\Gamma, S)(n) = \#\{\gamma \in \Gamma;\ |\gamma| \leq n\}\,.$$

One is often interested only in the type of the growth function. For instance, we say that the group Γ is of polynomial growth if there exist constants c and D such that

$$c^{-1} n^D \leq \beta(\Gamma, S)(n) \leq c n^D\,.$$

The exponent D does not depend on the set of generators S. If the growth function is bounded by a polynomial, it is known (see [6]) that Γ is of polynomial growth and D is an integer. For a group of polynomial growth with the exponent D, it is known (see [19]) that there exists a constant c such that

$$c^{-1} n^{-\frac{D}{2}} \leq P^{2n}(id, id) \leq c n^{-\frac{D}{2}}\,, \tag{5}$$

where $P^{2n}(id, id)$ is the probability of the return to the identity element of the simple random walk after $2n$ steps.

It seems natural to define a generalized growth function, using an eigenfunction of P. Let f be a positive eigenfunction of P corresponding to the eigenvalue equal to the norm of P, i.e.

$$Pf = \|P\| f\,.$$

The generalized growth function $\beta(\Gamma, S, f)$ associates to each positive integer n the number

$$\beta(\Gamma, S, f)(n) = \sum_{\gamma \in \Gamma, |\gamma| \leq n} f^2(\gamma),$$

i.e., each element in the ball of radius n is counted with weight f^2.

Let us compute the generalized growth function in a particular case. Let P be the simple random walk operator on the free group with the standard set of generators of cardinality k as in Section 4.1. Let g be the unique radial eigenfunction of P corresponding to the eigenvalue $\|P\|$ and such that $g(id) = 1$. Explicitly we have:

$$g(\gamma) = \left(\frac{k-2}{k}|\gamma| + 1\right)\left(\frac{1}{\sqrt{k-1}}\right)^{|\gamma|}.$$

Then we have

$$\sum_{\gamma \in \Gamma, |\gamma| \leq n} g^2(\gamma) =$$

$$n^3\left(\frac{k^2-4k+4}{3k^2-3k}\right) + n^2\left(\frac{3k^2-8k+4}{2k^2-2k}\right) + n\left(\frac{7k^2-16k+4}{6k^2-6k}\right) + 1.$$

This shows that the generalized growth is like n^3. In particular the sequence of balls is a generalized Følner sequence.

By analogy to (5) we conjecture that the fact that the generalized growth function for the free groups is like n^3 explains that for the free groups one has (see [16]):

$$c^{-1}\lambda^{2n}n^{-\frac{3}{2}} \leq P^{2n}(id, id) \leq c\lambda^{2n}n^{-\frac{3}{2}},$$

where c is a constant and λ is the norm of P.

4.2 Free products of finite groups

Random walks on free products of finite groups were already considered in [1], [3], [17] and [21].

Let us consider the group $\mathbf{Z}_m \star \mathbf{Z}_n$ with the following generating set:

- if $m \neq 2$ we take $\{\pm 1\}$ as generators of $\mathbf{Z}_m = \{0, 1, \ldots, m-1\}$;
- we take $\{1\}$ as a generator of $\mathbf{Z}_2 = \{0, 1\}$.