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**Artikel:** CLASS NUMBER FORMULAE FOR IMAGINARY QUADRATIC NUMBER FIELDS  $\mathbb{Q}(\sqrt{-n})$  WITH  $n$  SQUAREFREE AND  $n \equiv 1 \pmod{4}$  OR  $n \equiv 2 \pmod{4}$   
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**THEOREM.** *Let  $n$  be a positive, squarefree integer with either  $n \equiv 1 \pmod{4}$  or  $n \equiv 2 \pmod{4}$  and with  $(a, 2n) = 1$ , and let  $j$  be a positive integer with  $(j, 2n) = 1$  and  $1 \leq j \leq n$ . Then if  $\left(\frac{-4n}{j}\right) = +1$ , we have*

$$h(-n) = \frac{1}{2} \sum_{i=0}^{\frac{j-1}{2}} \sum_{a=\left[\frac{4in}{j}\right]+1}^{\left[\frac{(4i+2)n}{j}\right]} \left(\frac{-4n}{a}\right),$$

and if  $\left(\frac{-4n}{j}\right) = -1$ , then we have

$$h(-n) = \frac{1}{2} \sum_{i=1}^{\frac{j-1}{2}} \sum_{a=\left[\frac{(4i-2)n}{j}\right]+1}^{\left[\frac{4in}{j}\right]} \left(\frac{-4n}{a}\right).$$

If  $j = 1$ , the result is due to Dirichlet [3], [4]. We illustrate the theorem when  $n = 13$  and  $j = 3$ . Then  $\left(\frac{-52}{3}\right) = \left(\frac{-1}{3}\right) = -1$ . Thus

$$h(-13) = \frac{1}{2} \sum_{a=9}^{17} \left(\frac{-52}{a}\right).$$

Now  $\left(\frac{-52}{9}\right) = \left(\frac{-52}{11}\right) = \left(\frac{-52}{15}\right) = \left(\frac{-52}{17}\right) = +1$ , and so  $h(-13) = \frac{1}{2}(4) = 2$ . The study of class numbers relating values of the Jacobi symbol  $\left(\frac{a}{n}\right)$  to  $h(-n)$  when  $n \equiv 3 \pmod{4}$  in subintervals other than  $(0, \frac{n}{2})$  has been given by numerous authors. These include among others, Berndt [1], Berndt and Chowla [2], Dirichlet [3]–[4], Holden [5]–[11], Hudson and Williams [12], Johnson and Mitchell [13], Karpinski [14], and Lerch [15]–[16]. A partial summary of these results appears in [12].

## 2. PROOF OF THE THEOREM

We first note that  $j$  is an odd, positive integer with  $(j, n) = 1$ . We write

$$\sum_{\substack{a=1 \\ (a, 2n)=1}}^{2n-1} \left(\frac{-4n}{a}\right) = \sum_{r=0}^{j-1} S_r$$

where

$$S_r = \sum_{\substack{a=1 \\ a \equiv r \pmod{j} \\ (a, 2n)=1}}^{2n-1} \left( \frac{-4n}{a} \right).$$

If  $1 \leq r \leq j-1$ , then there exists a unique integer  $k$  such that  $1 \leq k \leq j-1$  and  $2kn \equiv r \pmod{j}$  because  $(j, n) = 1$ . If  $a \equiv r \pmod{j}$  with  $1 \leq a \leq 2n-1$  and  $(a, 2n) = 1$ , then we observe that  $2kn - a \equiv 0 \pmod{j}$ . Now

$$\left( \frac{-4n}{a} \right) = \left( \frac{-4n}{2kn - a} \right)$$

if  $k$  is odd, and

$$\left( \frac{-4n}{a} \right) = - \left( \frac{-4n}{2kn - a} \right)$$

if  $k$  is even. Thus,

$$\begin{aligned} S_r &= \sum_{\substack{a=(2k-2)n \\ a \equiv 0 \pmod{j} \\ (a, 2n)=1}}^{2kn} \left( \frac{-4n}{2kn - a} \right) \\ &= \pm \sum_{\substack{a=(2k-2)n \\ a \equiv 0 \pmod{j} \\ (a, 2n)=1}}^{2kn} \left( \frac{-4n}{a} \right) \\ &= \pm \left( \frac{-4n}{j} \right) \sum_{\substack{a=[\frac{(2k-2)n}{j}] + 1 \\ (a, 2n)=1}}^{[\frac{2kn}{j}]} \left( \frac{-4n}{a} \right) \end{aligned}$$

where the plus sign holds if  $k$  is odd and the minus sign holds if  $k$  is even. Thus we have for each  $j$ ,

$$\begin{aligned} 0 &= \left( \frac{-4n}{j} \right) \sum_{\substack{k=1 \\ (k, 2)=2}}^j \sum_{\substack{a=[\frac{(2k-2)n}{j}] + 1 \\ (a, 2n)=1}}^{[\frac{2kn}{j}]} \left( \frac{-4n}{a} \right) \\ &+ \left( \frac{-4n}{j} \right) \sum_{\substack{k=1 \\ (k, 2)=1}}^j \sum_{\substack{a=[\frac{(2k-2)n}{j}] + 1 \\ (a, 2n)=1}}^{[\frac{2kn}{j}]} \left( \frac{-4n}{a} \right) - \sum_{\substack{a=1 \\ (a, 2n)=1}}^{2n-1} \left( \frac{-4n}{a} \right). \end{aligned}$$

It then follows that

$$0 = -2 \sum_{\substack{k=1 \\ (k,2)=2}}^j \sum_{\substack{a=\left[\frac{(2k-2)n}{j}\right]+1 \\ (a,2n)=1}}^{\left[\frac{2kn}{j}\right]} \left(\frac{-4n}{a}\right)$$

if  $\left(\frac{-4n}{j}\right) = +1$ , and

$$0 = -2 \sum_{\substack{k=1 \\ (k,2)=1}}^j \sum_{\substack{a=\left[\frac{(2k-2)n}{j}\right]+1 \\ (a,2n)=1}}^{\left[\frac{2kn}{j}\right]} \left(\frac{-4n}{a}\right)$$

if  $\left(\frac{-4n}{j}\right) = -1$ . In the case that  $\left(\frac{-4n}{j}\right) = +1$ , we are only considering those  $k$  which are even, and so we may write  $k = 2i$ . In the case that  $\left(\frac{-4n}{j}\right) = -1$ , we are only considering those  $k$  which are odd, and so we may write  $k = 2i + 1$ .

Thus we have proven that for each  $j$ ,

$$0 = \sum_{i=1}^{\left[\frac{j}{2}\right]} \sum_{\substack{a=\left[\frac{(4i-2)n}{j}\right]+1 \\ (a,2n)=1}}^{\left[\frac{4in}{j}\right]} \left(\frac{-4n}{a}\right)$$

if  $\left(\frac{-4n}{j}\right) = +1$ , and

$$0 = \sum_{i=0}^{\left[\frac{j}{2}\right]} \sum_{\substack{a=\left[\frac{4in}{j}\right]+1 \\ (a,2n)=1}}^{\left[\frac{(4i+2)n}{j}\right]} \left(\frac{-4n}{a}\right)$$

if  $\left(\frac{-4n}{j}\right) = -1$ . These subintervals clearly cover  $[1, 2n - 1]$  and are non-overlapping. Now Dirichlet [3], [4] showed that

$$\sum_{\substack{a=1 \\ (a,2n)=1}}^{2n-1} \left(\frac{-4n}{a}\right) = 2h(-n).$$

It follows at once that

$$h(-n) = \frac{1}{2} \sum_{i=0}^{\frac{j-1}{2}} \sum_{a=\lfloor \frac{4i}{j} \rfloor + 1}^{\lfloor \frac{(4i+2)n}{j} \rfloor} \left( \frac{-4n}{a} \right)$$

if  $\left( \frac{-4n}{j} \right) = +1$ , and

$$h(-n) = \frac{1}{2} \sum_{i=1}^{\frac{j-1}{2}} \sum_{a=\lfloor \frac{(4i-2)n}{j} \rfloor + 1}^{\lfloor \frac{4in}{j} \rfloor} \left( \frac{-4n}{a} \right)$$

if  $\left( \frac{-4n}{j} \right) = -1$ .

### 3. REMARKS

In Bruce Berndt's paper "Classical Theorems on Quadratic Residues" [1], he uses the following notation:

$$S_{ji} = \sum_{\frac{(i-1)k}{j} < n < \frac{ik}{j}} \chi(n).$$

Using this notation, we can rewrite the class number formulae as follows:

1. If  $\left( \frac{-4n}{j} \right) = +1$ , then we have

$$h(-n) = \frac{1}{2} \sum_{i=0}^{\frac{j-1}{2}} S_{j,2i+1}.$$

2. If  $\left( \frac{-4n}{j} \right) = -1$ , then we have

$$h(-n) = \frac{1}{2} \sum_{i=1}^{\frac{j-1}{2}} S_{j,2i}.$$

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