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*Proof.* Let  $\varphi$  be an automorphism of  $X$ . If  $\varphi$  fixes a point  $p$  of  $\Lambda_i^*$ , then  $p$  can be chosen as a vertex or a midpoint of an edge. If  $p$  is a vertex, then the preimage  $X'$  of  $p$  under  $r_i$  is a closed and convex subcomplex of  $X$ . If  $p$  is the midpoint of an edge,  $X'$  is a hyperspace and as a union of walls, carries a natural cubical structure. In either case,  $X'$  is a closed, convex and  $\varphi$ -invariant subset of  $X$ , and therefore  $\varphi$  is semisimple if and only if the restriction  $\varphi|_{X'}$  is semisimple. Since moreover  $X'$  is a simply connected folded cubical chamber complex of nonpositive curvature and of dimension lower than  $X$ , we can assume by induction on  $\dim X$  that the action of  $\varphi$  on all the trees  $\Lambda_i^*$  is axial.

Let  $a_i$  be an axis of  $\varphi$  in  $\Lambda_i^*$  (unique up to parameter). Let  $X_i = r_i^{-1}(a_i)$ . Since  $r_i$  is surjective,  $X_i$  is non-empty. Furthermore,  $X_i$  is a closed, convex and  $\varphi$ -invariant subcomplex of  $X$ .

Set  $Y_1 := X_1$ . The image of  $Y_1$  under  $r_2$  is path connected and  $\varphi$ -invariant, hence contains  $a_2$ . Let  $Y_2 = Y_1 \cap X_2$ . Then  $Y_2$  is non-empty, closed, convex and  $\varphi$ -invariant. By induction we get that  $Y = X_1 \cap \dots \cap X_n$  is a non-empty, closed, convex and  $\varphi$ -invariant subcomplex of  $X$ . It is then sufficient to prove semisimplicity for the restriction  $\varphi|_Y$ . Note that  $Y = r^{-1}(F)$ , where  $F \cong \mathbf{R}^n$  is the flat

$$F = \{(a_1(t_1), \dots, a_n(t_n)) \mid t_i \in \mathbf{R}\}$$

in the product of trees. Now  $\varphi$  operates as a translation on  $F$ , hence the displacement of  $\varphi$  on  $F$  is constant, say  $= \delta$ . Since  $r$  is injective, we can consider  $Y$  as a closed subcomplex of  $F$ , namely a union of chambers. The metric on  $Y$  is the induced path metric. It follows easily that there are only finitely many possible values for the distance in  $Y$  from a point  $x$  to its image  $\varphi x$ , if the location of  $x$  in its chamber is given.  $\square$

#### 4. NONEXISTENCE OF FREE SUBGROUPS

In this section we discuss the proof of Theorem 2 of the introduction. We assume throughout this section that  $X$  is a simply connected folded cubical chamber complex of nonpositive curvature and that  $\Gamma \subset \text{Aut}(X)$  is a group that preserves the folding of  $X$  (this can be always assumed by passing to a finite index normal subgroup if necessary) and does not contain a free nonabelian subgroup acting freely on  $X$ . By equivariance of the maps  $r_i$ , the same holds for the actions of  $\Gamma$  on the trees  $\Lambda_i^*$ . Up to a subgroup of index two, there are three possibilities for each particular  $i$  [PV]:

- (0)  $\Gamma$  fixes a point of  $\Lambda_i^*$  ;
- (1)  $\Gamma$  fixes no point of  $\Lambda_i^*$ , but precisely one end of  $\Lambda_i^*$  ;
- (2)  $\Gamma$  fixes no point of  $\Lambda_i^*$ , but precisely two ends of  $\Lambda_i^*$ .

Thus by passing to a subgroup of  $\Gamma$  of index at most  $2^n$ , we can assume that the above three alternatives hold for all  $i$ . Corresponding to the alternative, we say that  $i$  is an *index of type 0, 1 or 2* respectively.

We first construct a homomorphism  $h = (h_1, \dots, h_n): \Gamma \rightarrow \mathbf{Z}^n$  as claimed. If  $\Gamma$  fixes a point of  $\Lambda_i^*$ , we define  $h_i$  to be the trivial homomorphism. If  $\Gamma$  does not fix a point of  $\Lambda_i^*$ , we let  $\omega_i$  be the end or one of the two ends of  $\Lambda_i^*$  fixed by  $\Gamma$ . The Busemann function  $b_i: \Lambda_i^* \rightarrow \mathbf{R}$  at  $\omega_i$  is well defined up to an additive constant (see [Ba], Section 1 of Chapter II). Since  $\Gamma$  fixes  $\omega_i$ ,

$$h_i(\phi) := b_i(\phi p) - b_i(p), \quad p \in \Lambda_i^*,$$

is a well defined homomorphism  $h_i: \Gamma \rightarrow \mathbf{Z}$ , called the *Busemann homomorphism*. Note that  $h_i$  is integer valued since  $\Lambda_i^*$  is a simplicial tree and  $\Gamma$  acts by automorphisms. This completes the definition of  $h = (h_1, \dots, h_n)$ . We set

$$\Delta_i = \ker h_i \quad \text{and} \quad \Delta = \bigcap \Delta_i = \ker h.$$

PROPOSITION 4.1.  $\Delta$  consists precisely of the elliptic elements of  $\Gamma$ .

*Proof.* If the action of  $\Gamma$  on  $\Lambda_i^*$  has a fixed point, then any  $\phi \in \Gamma$  is elliptic on  $\Lambda_i^*$  and  $\Delta_i = \Gamma$ . If  $\Gamma$  does not have a fixed point in  $\Lambda_i^*$ , but fixes a point  $\xi_i \in \Lambda_i^*(\infty)$  and  $\phi \in \Gamma$  is axial on  $\Lambda_i^*$ , then  $\xi_i$  is an end point of the axis of  $\phi$ . Then  $h_i(\phi) \neq 0$ . Hence by Proposition 3.5, any  $\phi \in \Delta$  is elliptic on  $X$ . Conversely, if  $\phi \in \Gamma$  is elliptic on  $X$ , then  $\phi \in \Delta$ .  $\square$

For the proof of the other assertions of Theorem 2 we need some more preparations.

LEMMA 4.2. Let  $\Lambda$  be a simplicial tree on which  $\Gamma$  acts by automorphisms. Suppose  $\Delta$  fixes a point of  $\Lambda$ . Then either  $\Gamma$  fixes a point of  $\Lambda$  or exactly two points in  $\Lambda(\infty)$ .

*Proof.* Since  $\Delta$  is a normal subgroup of  $\Gamma$ , the set  $\Phi$  of fixed points of  $\Delta$  is  $\Gamma$ -invariant. Now  $\Phi$  is a subtree of  $\Lambda$ , hence we can assume  $\Phi = \Lambda$ . Then the quotient action by  $\Gamma/\Delta$  on  $\Lambda$  is well defined.

Suppose that  $\Gamma/\Delta$  contains an element  $\phi$  which is axial on  $\Lambda$ . Since  $\Gamma/\Delta$  is abelian, it leaves the unique axis of  $\phi$  invariant and fixes the endpoints of the axis.

Suppose now that all elements of  $\Gamma/\Delta$  are elliptic on  $\Lambda$ . Let  $\phi_1, \dots, \phi_k$  be a system of generators. The set of fixed points of  $\phi_1$  is a  $\Gamma/\Delta$ -invariant subtree. Replacing  $\Lambda$  by this subtree, we can assume that  $\phi_1 = \text{id}_\Lambda$ . The quotient of  $\Gamma/\Delta$  by the subgroup generated by  $\phi_1$  is abelian and has a system of  $k-1$  generators. Induction on  $k$  shows that  $\Gamma$  has a fixed point.  $\square$

If  $i$  is an index of type 0 and  $p \in \Lambda_i^*$  a fixed point, then  $X' := r_i^{-1}(p) \subset X$  is closed, convex and  $\Gamma$ -invariant. In particular,  $X'(\infty) \subset X(\infty)$  is  $\Gamma$ -invariant. Although  $X'$  is not a subcomplex if  $p$  is not a vertex, it is parallel to the walls with label  $i$  in the chambers it intersects. Hence we obtain a natural cubical structure on  $X'$  with a folding onto an  $(n-1)$ -cube, and  $\Gamma$  preserves this cubical structure and folding. Hence by passing to such subspaces if necessary, we can assume that no indices of type 0 occur.

Let  $i$  be an index of type 2. Let  $\alpha_i, \omega_i \in \Lambda_i^*(\infty)$  be the fixed points of  $\Gamma$  and  $\sigma_i$  the unit speed geodesic from  $\alpha_i$  to  $\omega_i$ . Then  $\sigma_i$  is  $\Gamma$ -invariant and  $\Delta_i = \text{Stab}(\sigma_i(t))$  for all  $t \in \mathbf{R}$ . Hence  $X' = r_i^{-1}(\text{im } \sigma_i)$  is a closed, convex and  $\Gamma$ -invariant subcomplex of  $X$ . Hence by passing to such subspaces if necessary, we can assume that  $\Lambda_i^* = \text{im } \sigma_i \cong \mathbf{R}$  for all indices  $i$  of type 2.

**PROPOSITION 4.3.** *If there are no indices of type 1, then there is a  $\Gamma$ -invariant convex subset  $E \subset X$  isometric to a Euclidean space of dimension  $k \in \{0, \dots, n\}$  and an exact sequence*

$$0 \rightarrow \Delta \rightarrow \Gamma \rightarrow \mathbf{Z}^k \rightarrow 0$$

*such that  $\Delta$  fixes  $E$  pointwise and such that the quotient  $\Gamma/\Delta \cong \mathbf{Z}^k$  acts on  $E$  as a cocompact lattice of translations.*

*Proof.* After reductions as above we can assume that all indices are of type 2, that  $\Lambda_i^* \cong \mathbf{R}$  for all  $i$  and that  $\Delta$  fixes each point of  $\prod \Lambda_i^*$ . Since  $r$  is an injection,  $\Delta$  fixes each point of  $X$ .

The image  $\text{im } h$  of the homomorphism  $h$  is a subgroup of the group  $\mathbf{Z}^n$ , hence it is isomorphic to  $\mathbf{Z}^k$  for some  $k \leq n$ . Thus we may identify the quotient group  $\Gamma/\Delta$  with  $\mathbf{Z}^k$ . Consider the quotient action of  $\mathbf{Z}^k = \Gamma/\Delta$  on  $X$ , which is well defined since  $\Delta$  acts trivially on  $X$ . This action is free and the elements are semisimple by Proposition 3.6. Applying the Flat Torus Theorem, see [CE] and [BH], we get that there exists a  $\mathbf{Z}^k$ -invariant convex subspace  $E \subset X$ , isometric to  $k$ -dimensional Euclidean space, such that  $\mathbf{Z}^k$  acts on it as a cocompact lattice of translations.  $\square$

We now discuss the more difficult case that indices of type 1 occur. As explained above, we can assume that no indices of type 0 occur and that  $\Lambda_i^* \cong \mathbf{R}$  for all indices of type 2.

Choose a vertex  $x_0 \in X$  as an origin. For indices of type 2 choose the parameter on the above geodesics  $\sigma_i$  such that  $\sigma_i(0) = r_i(x_0)$ . For indices of type 1 we denote by  $\omega_i \in \Lambda_i^*(\infty)$  the corresponding fixed point. For these indices, we let  $\sigma_i: [0, \infty) \rightarrow \Lambda_i^*$  be a unit speed geodesic ray with  $\sigma_i(0) = r_i(x_0)$  and  $\sigma_i(\infty) = \omega_i$ .

We set  $F = \text{im } \sigma_1 \times \cdots \times \text{im } \sigma_n$ . Note that  $F$  is a closed and convex subspace of  $\prod \Lambda_i^*$ . We also define a geodesic ray

$$\sigma: [0, \infty) \rightarrow F \quad \text{by} \quad \sigma(t) = (\sigma_1(t), \dots, \sigma_n(t)).$$

By construction,  $\sigma(0) = r(x_0)$ .

LEMMA 4.4.  *$\text{Stab}(\sigma_i(t)) \rightarrow \Delta_i$  and  $\text{Stab}(\sigma(t)) \rightarrow \Delta$  as  $t \rightarrow \infty$ , where the limit of groups is understood as the union of increasing family.*

*Proof.* Let  $\phi \in \Delta_i$ . Then  $\phi$  fixes  $\omega_i = \sigma_i(\infty)$ . Therefore  $\phi \circ \sigma_i$  is asymptotic to  $\sigma_i$ . Now  $\Lambda_i^*$  is a tree, hence  $\phi \circ \sigma_i(t) = \sigma_i(t + c)$  for all  $t$  sufficiently large, where  $c$  is some constant independent of  $t$ . Since  $\phi \in \Delta_i$ ,  $c = 0$  and therefore  $\phi \in \text{Stab}(\sigma_i(t))$  for all  $t$  sufficiently large.  $\square$

COROLLARY 4.5. *There exists a sequence  $(x_m)$  in  $X$  such that  $\text{Stab}(x_m) \rightarrow \Delta$ .*

*Proof.* We observe that  $\text{Stab}(x) \subset \Delta$  for all  $x \in X$ . Now the assertion follows immediately from Proposition 3.5 and Lemma 4.4.  $\square$

LEMMA 4.6. *If the group  $\Gamma$  fixes precisely one point  $\omega_i \in \Lambda_i^*(\infty)$ , then  $\Delta \cap \text{Stab}(\sigma_i(t))$  has infinitely many jumps as  $t \rightarrow \infty$ .*

*Proof.* Let  $\phi \in \Delta \subset \Delta_i$ . By Lemma 4.4 there is  $t_\phi \geq 0$  such that  $\phi \in \text{Stab}(\sigma_i(t))$  for all  $t \geq t_\phi$ . Hence if  $\Delta \cap \text{Stab}(\sigma_i(t)) = \Delta \cap \text{Stab}(\sigma_i(t'))$  for all  $t, t'$  sufficiently large, then  $\Delta \subset \text{Stab}(\sigma_i(t))$  for all  $t$  sufficiently large. By Lemma 4.2,  $\Gamma$  either fixes a point of  $\Lambda_i^*$ , which is excluded by our reductions above, or  $\Gamma$  fixes exactly two points of  $\Lambda_i^*(\infty)$ , which is in contradiction to the assumption.  $\square$

LEMMA 4.7. *Let  $(x_m)$  be a sequence in  $X$  such that  $\text{Stab}(x_m) \rightarrow \Delta$  and  $\gamma_m: [0, s_m] \rightarrow X$  be the unit speed geodesic from  $x_0$  to  $x_m$ , where  $s_m = d(x_0, x_m)$ . Then given a constant  $t_0 > 0$ , there exists  $m_0$  such that  $s_m \geq t_0$  and  $r \circ \gamma_m([0, t_0]) \in F$  for all  $m \geq m_0$ .*

*Proof.* For those  $i$  for which  $\Gamma$  fixes exactly one point  $\omega_i \in \Lambda_i^*(\infty)$  we choose  $\phi_i \in \Delta$  such that  $\phi_i \notin \text{Stab}(\sigma_i(t))$  for  $t \leq t_0$ , see Lemma 4.6. By assumption, there is  $m_0$  such that  $\phi_i \in \text{Stab}(x_m)$  for all  $m \geq m_0$  and all such  $i$ . Now  $r_i \circ \gamma_m$  is a monotonic curve in  $\Lambda_i^*$  from  $\sigma_i(0) = r_i(x_0)$  to  $r_i(x_m)$ . By equivariance of  $r_i$ ,  $\phi_i \in \text{Stab}(r_i(x_m))$  for all  $m \geq m_0$ . On the other hand,  $r_i \circ \sigma$  has speed  $\leq 1$ , hence by the choice of  $t_0$ ,  $s_m \geq t_0$  and  $r_i(\gamma_m(t)) \in \sigma_i([0, t_0])$  for  $0 \leq t \leq t_0$ .

The claim follows since the image of  $r_i$  is  $\sigma_i$  for those  $i$  for which  $\Gamma$  fixes exactly two ends of  $\Lambda_i^*$ .  $\square$

LEMMA 4.8. *Given  $\phi \in \Gamma$ , there is a constant  $c = c_\phi$  such that  $d(\phi(p), p) \leq c$  for all  $p \in F$ .*

*Proof.* We show that  $d_i(\phi(p), p) \leq c_i$  for each point  $p$  in the image of  $\sigma_i$ . This is clear for those indices  $i$  for which  $\Gamma$  fixes exactly two ends of  $\Lambda_i^*$ . Consider some other index  $i$ . Then  $\sigma_i$  is defined on  $[0, \infty)$ .

If  $\phi$  is elliptic on  $\Lambda_i^*$ , then  $\phi \in \Delta_i$ . By Lemma 4.4, there exists a constant  $t_\phi$  such that  $\phi$  fixes  $\sigma_i(t)$  for all  $t \geq t_\phi$ . We conclude that  $d_i(\phi(p), p) \leq 2t_\phi$  for each point  $p$  in the image of  $\sigma_i$ .

We assume now that  $\phi$  is axial on  $\Lambda_i^*$  and let  $\rho$  be an axis of  $\phi$  in  $\Lambda_i^*$ . We parametrize  $\rho$  such that  $\rho(\infty) = \omega_i$ . Since  $\Lambda_i^*$  is a tree and  $\sigma_i(\infty) = \rho(\infty)$ , we can actually choose the parameter such that  $\sigma_i(t) = \rho(t)$  for all  $t \geq t_\phi$ , where  $t_\phi$  is an appropriate constant. Now  $\phi(\rho(t)) = \rho(t + \tau)$  for some constant  $\tau$  independent of  $t$ . We conclude that  $d_i(\phi(p), p) \leq 2t_\phi + \tau$  for each point  $p$  in the image of  $\sigma_i$ .  $\square$

PROPOSITION 4.9. *Suppose that indices of type 1 occur. Then*

- (1)  $\Delta$  does not fix a point of  $X$ ;
- (2)  $\Gamma$  fixes a point in  $X(\infty)$ . More precisely, if  $(x_m)$  is a sequence in  $X$  such that  $\text{Stab}(x_m) \rightarrow \Delta$ , then after passing to a subsequence if necessary,  $(x_m)$  converges to a fixed point  $\xi \in X(\infty)$  of  $\Gamma$ .

*Proof.* The first assertion is an immediate consequence of Lemma 4.7. As for the proof of the second assertion, let  $(x_m)$  be a sequence in  $X$  with  $\text{Stab}(x_m) \rightarrow \Delta$ . Let  $\gamma_m: [0, s_m] \rightarrow X$  be the unit speed geodesic from  $x_0$  to  $x_m$  as in Lemma 4.7. Note that  $r \circ \gamma_m$  is a sequence of unit speed curves (with respect to the metric  $d_{(2)}$ , for which  $r$  restricted to any chamber of  $X$  is an isometry) in  $\coprod \Lambda_i^*$ . For each constant  $t_0 > 0$ ,  $r \circ \gamma_m([0, t_0])$  is contained in  $F$  for all  $m$  sufficiently large. Now  $F$  is locally compact, hence a subsequence of

the sequence of curves  $r \circ \gamma_m$  converges locally uniformly. By Proposition 3.4, the corresponding subsequence of the sequence of unit speed geodesics  $\gamma_m$  converges locally uniformly. By definition, this means that the corresponding subsequence of  $(x_m)$  converges to a point  $\xi \in X(\infty)$ .

Let  $\phi \in \Gamma$  and choose  $c = c_\phi$  as in Lemma 4.8. Let  $t_0 > 0$  be given. By Lemma 4.8 we have  $r \circ \gamma_m(t_0) \in F$  for all  $m \geq m_0$ . By Proposition 3.4 and Lemma 4.8, we have  $d(\phi(\gamma_m(t_0)), \gamma_m(t_0)) \leq \sqrt{nc_\phi}$  for all such  $m$ . Now  $c_\phi$  is independent of  $t_0$ , hence  $\phi(\xi) = \xi$ .  $\square$

We now complete the proof of Theorem 2 of the introduction. By Proposition 4.1,  $\Delta \cong \ker h$  consists precisely of the elliptic elements of  $\Gamma$ . If indices of type 1 do not occur, then Proposition 4.3 applies: If  $k = 0$ , then  $\Gamma \cong \Delta$  fixes a point of  $X$  and possibility (1) holds. If  $k > 0$ , then possibility (2) holds. If indices of type 1 occur, then possibility (3) holds by Proposition 4.9 and Corollary 4.5. Note that  $\text{Stab}(x) \neq \Delta$  for any  $x \in X$  in this case since  $\Delta$  would have a fixed point otherwise.

## 5. PARALLEL TRANSPORT IN A CUBICAL MANIFOLD AND THE PROOF OF THEOREM 3

Let  $X$  be a cubical manifold of dimension  $n$ . Given two chambers  $P$  and  $Q$  in  $X$  with a common face of dimension  $n - 1$ , we define  $t_{PQ}: P \rightarrow Q$  to be the *translation* which moves each point  $p$  of  $P$  along the unit geodesic segment starting at  $p$  and orthogonal to the common  $(n - 1)$ -face of  $P$  to the end point in  $Q$ . The map  $t_{PQ}$  is an isomorphism and isometry of  $P$  with  $Q$ . Given a gallery  $\pi = (P_1, \dots, P_n)$  in  $X$ , the *parallel transport* along  $\pi$  is the isomorphism  $t_\pi: P_1 \rightarrow P_n$  given by

$$t_\pi := t_{P_{n-1}P_n} \circ \cdots \circ t_{P_2P_3} \circ t_{P_1P_2}.$$

**LEMMA 5.1.** *Let  $X$  be a simply connected cubical manifold and assume that the number of chambers adjacent to each face of codimension 2 in  $X$  is divisible by 4. Then for any two chambers  $P$  and  $Q$  in  $X$ , the parallel transport  $t_\pi$  along a gallery  $\pi$  connecting  $P$  and  $Q$  is independent of  $\pi$ .*

*Proof.* It is enough to show that the parallel transport along any closed gallery is the identity. Let  $\pi$  be such a gallery with initial and final chamber  $P$ .