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Note that if $\mathcal{X}$ is not quasi-transitive, a somewhat weaker result holds [Kit98, §7.1]: if $\mathcal{X}$ is transient or null-recurrent then the common limsup is 0. If $\mathcal{X}$ is positive-recurrent then the limsups are normalized coefficients of $\mathcal{X}$'s Perron-Frobenius eigenvector. Lemma 3.9 is not true for arbitrary d-regular graphs: consider for instance the graph $\mathcal{X}_3$ described above. Its circuit series G_3, given in (3.3), has radius of convergence $1/\beta = 2/7$, and one easily checks that all its coefficients g_n satisfy $g_n/\beta^n \geq 1/2$.

We obtain the following characterization of rational series:

THEOREM 3.10. *For regular quasi-transitive connected graphs $\mathcal{X}$, the following are equivalent:*

1. *$\mathcal{X}$ is finite;*
2. *$G(t)$ is a rational function of t;*
3. *$F(t)$ is a rational function of t, and $\mathcal{X}$ is not an infinite tree.*

Proof. By Corollary 2.7, Statement 1 implies the other two. By Corollary 2.6, and a computation on trees done in Section 7.3 to deal with the case $F(t) = 1$, Statement 2 implies 3. It remains to show that Statement 3 implies 1.

Assume that $F(t) = \sum f_n t^n$ is rational, not equal to 1. As the f_n are positive, F has a pole, of multiplicity m, at $1/\alpha$. There is then a constant $a > 0$ such that $f_n > a\binom{n}{m-1}\alpha^n$ for infinitely many values of n [GKP94, page 341]. It follows by Lemma 3.9 that $m = 1$ and the graph $\mathcal{X}$ is finite, of cardinality at most $1/a$. □

It is not known whether the same holds for regular, or even arbitrary connected graphs. Certainly an altogether different proof would be needed.

3.3 APPLICATION TO LANGUAGES

Let S be a finite set of cardinality d and let $\bar{\cdot}$ be an involution on S. A *word* is an element w of the free monoid S^*. A *language* is a set L of words. The language L is called *saturated* if for any $u, v \in S^*$ and $s \in S$ we have

$$uv \in L \Longleftrightarrow us\bar{s}v \in L;$$

that is to say, L is stable under insertion and deletion of subwords of the form $s\bar{s}$. The language L is called *desiccated* if no word in L contains a subword of the form $s\bar{s}$. Given a language L we may naturally construct its *saturation*

$\langle L \rangle$, the smallest saturated language containing L, and its *desiccation* $\widehat{L}$, the largest desiccated language contained in L.

Let Σ be the monoid defined by generators S and relations $s\bar{s} = 1$ for all $s \in S$:

$$\Sigma = \langle S \mid s\bar{s} = 1 \quad \forall s \in S \rangle . \tag{3.4}$$

This is a free product of free groups and order-two groups; if $\bar{\cdot}$ is fixed-point-free, Σ is a free group. Write ϕ for the canonical projection from S^* to Σ. Let $\mathbf{k} = \mathbf{Z}[\Sigma]$ be its monoid ring. Then given a language $L \subset S^*$ we may define its *growth series* $\Theta(L)$ as

$$\Theta(L) = \sum_{w \in L} w^{\phi} t^{|w|} \in \mathbf{k}[[t]] .$$

This notion of growth series with coefficients was introduced by Fabrice Liardet in his doctoral thesis [Lia96], where he studied *complete growth functions* of groups.

THEOREM 3.11. *For any language L there holds*

$$\frac{\Theta(\widehat{L})(t)}{1-t^2} = \frac{\Theta(\langle L \rangle)\left(\frac{t}{1+(d-1)t^2}\right)}{1+(d-1)t^2} , \tag{3.5}$$

where $d = |S|$.

Proof. For any language there exists a unique minimal (possibly infinite) automaton recognising it ([Eil74, §III.5] is a good reference). Let $\mathcal{X}$ be the minimal automaton recognising $\langle L \rangle$. Recall that this is a graph with an initial vertex $\star$, a set of terminal vertices T and a labelling $\ell' \colon E(\mathcal{X}) \to S$ of the graph's edges such that the number of paths labelled w, starting at $\star$ and ending at a $\tau \in T$ is 1 if $w \in L$ and 0 otherwise. Extend the labelling ℓ' to a labelling $\ell \colon E(\mathcal{X}) \to \mathbf{k}[[t]]$ by

$$e^{\ell} = t \cdot (e^{\ell'})^{\phi} .$$

Because $\langle L \rangle$ is saturated, and $\mathcal{X}$ is minimal, $(\bar{e})^{\ell} = \overline{e^{\ell}}$; then $\widehat{L}$ is the set of labels on proper paths from $\star$ to some $\tau \in T$. Choosing in turn all $\tau \in T$ as $\dagger$, we obtain growth series F_{τ}, G_{τ} counting the formal sum of paths and proper paths from $\star$ to τ. It then suffices to write

$$\frac{\Theta(\widehat{L})(t)}{1-t^2} = \frac{\sum_{\tau \in T} F_{\tau}(t)}{1-t^2} = \frac{\sum_{\tau \in T} G_{\tau}\left(\frac{t}{1+(d-1)t^2}\right)}{1+(d-1)t^2} = \frac{\Theta(\langle L \rangle)\left(\frac{t}{1+(d-1)t^2}\right)}{1+(d-1)t^2} . \qquad \square$$

The following result is well-known:

THEOREM 3.12 (Müller & Schupp [MS81, MS83]). *Let Γ be a finitely generated group, presented as a quotient Σ/Ξ with Σ as in (3.4). Then $\Theta(\Xi)$ is an algebraic series (i.e. satisfies a polynomial equation over $\mathbf{k}[t]$) if and only if Σ/Ξ is virtually free (i.e. has a normal subgroup of finite index that is free).*

It is not known whether there exists a non-virtually-free quasi-transitive graph whose circuit series (as defined in Corollary 2.6) is algebraic.

4. FIRST PROOF OF THEOREM 2.4

We now prove Theorem 2.4 using linear algebra. We first assume the graph has a finite number of vertices, for the computations refer to $\mathbf{k}$-matrices and $\mathbf{k}[[u]]$-matrices indexed by the graph's vertices. This proof is hinted at in Godsil's book as an exercise [God93, page 72]; it was also suggested to the author by Gilles Robert.

For all pairs of vertices $x, y \in V(\mathcal{X})$ let

$$\mathfrak{G}_{x,y}(\ell) = \sum_{\pi \in [x,y]} \pi^{\ell}, \qquad \mathfrak{F}_{x,y}(\ell) = \sum_{\pi \in [x,y]} u^{\mathrm{bc}(\pi)} \pi^{\ell}$$

be the path and enriched path series from x to y; for ease of notation we will leave out the labelling ℓ if it is obvious from the context. Let $\delta_{x,y}$ denote the Kronecker delta, equal to 1 if $x = y$ and 0 otherwise. For any $v \in \mathbf{k}$, let $[v]_x^y$ denote the $V(\mathcal{X}) \times V(\mathcal{X})$ matrix with zeros everywhere except at (x, y), where it has value v. Then

$$\mathfrak{G}_{x,y} = \delta_{x,y} + \sum_{e \in E(\mathcal{X}) : e^{\alpha} = x} e^{\ell} \mathfrak{G}_{e^{\omega},y}$$

so that if

$$A = \sum_{e \in E(\mathcal{X})} [e^{\ell}]_{e^{\alpha}}^{e^{\omega}}$$

be the adjacency matrix of $\mathcal{X}$, with labellings, then we have

$$(\mathfrak{G}_{x,y})_{x,y \in V(\mathcal{X})} = \frac{1}{1 - A},$$

an equation holding between $V(\mathcal{X}) \times V(\mathcal{X})$ matrices over $\mathbf{k}$.