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The following result is well-known:

THEOREM 3.12 (Müller & Schupp [MS81, MS83]). *Let* Γ *be a finitely generated group, presented as a quotient* Σ/Ξ *with* Σ *as in (3.4). Then* $\Theta(\Xi)$ *is an algebraic series (i.e. satisfies a polynomial equation over* $\mathbf{k}[t]$*) if and only if* Σ/Ξ *is virtually free (i.e. has a normal subgroup of finite index that is free).*

It is not known whether there exists a non-virtually-free quasi-transitive graph whose circuit series (as defined in Corollary 2.6) is algebraic.

4. FIRST PROOF OF THEOREM 2.4

We now prove Theorem 2.4 using linear algebra. We first assume the graph has a finite number of vertices, for the computations refer to $\mathbf{k}$-matrices and $\mathbf{k}[[u]]$-matrices indexed by the graph's vertices. This proof is hinted at in Godsil's book as an exercise [God93, page 72]; it was also suggested to the author by Gilles Robert.

For all pairs of vertices $x, y \in V(\mathcal{X})$ let

$$\mathfrak{G}_{x,y}(\ell) = \sum_{\pi \in [x,y]} \pi^{\ell}, \qquad \mathfrak{F}_{x,y}(\ell) = \sum_{\pi \in [x,y]} u^{\mathrm{bc}(\pi)} \pi^{\ell}$$

be the path and enriched path series from x to y; for ease of notation we will leave out the labelling ℓ if it is obvious from the context. Let $\delta_{x,y}$ denote the Kronecker delta, equal to 1 if $x = y$ and 0 otherwise. For any $v \in \mathbf{k}$, let $[v]_x^y$ denote the $V(\mathcal{X}) \times V(\mathcal{X})$ matrix with zeros everywhere except at (x, y), where it has value v. Then

$$\mathfrak{G}_{x,y} = \delta_{x,y} + \sum_{e \in E(\mathcal{X}) \,:\, e^{\alpha} = x} e^{\ell} \mathfrak{G}_{e^{\omega},y}$$

so that if

$$A = \sum_{e \in E(\mathcal{X})} [e^{\ell}]_{e^{\alpha}}^{e^{\omega}}$$

be the adjacency matrix of $\mathcal{X}$, with labellings, then we have

$$(\mathfrak{G}_{x,y})_{x,y \in V(\mathcal{X})} = \frac{1}{1 - A},$$

an equation holding between $V(\mathcal{X}) \times V(\mathcal{X})$ matrices over $\mathbf{k}$.

Similarly, letting $\mathfrak{F}_{x,e,y}$ count the paths from x to y that start with the edge e,

$$\mathfrak{F}_{x,y} = \delta_{x,y} + \sum_{e\in E(\mathcal{X}):e^\alpha=x} \mathfrak{F}_{x,e,y}\,,$$

$$\mathfrak{F}_{x,e,y} = e^\ell\left(\mathfrak{F}_{e^\omega,y} + (u-1)\mathfrak{F}_{e^\omega,\overline{e},y}\right)\,,$$

$$\mathfrak{F}_{e^\omega,\overline{e},y} = \overline{e}^\ell\left(\mathfrak{F}_{x,y} + (u-1)\mathfrak{F}_{x,e,y}\right)\,;$$

these last two lines solve to

$$\mathfrak{F}_{x,e,y} = \left(1-(u-1)^2(e\overline{e})^\ell\right)^{-1}\left(e^\ell\mathfrak{F}_{e^\omega,y} + (u-1)(e\overline{e})^\ell\mathfrak{F}_{x,y}\right)\,,$$

which we insert in the first line to obtain

$$K_x^{-1}\mathfrak{F}_{x,y} = \delta_{x,y} + \sum_{e\in E(\mathcal{X}):e^\alpha=x} \frac{e^\ell}{1-(u-1)^2(e\,\overline{e})^\ell}K_{e^\omega}\cdot K_{e^\omega}^{-1}\mathfrak{F}_{e^\omega,y}\,.$$

Thus if we let

$$e^{\ell'} = \frac{e^\ell}{1-(u-1)^2(e\,\overline{e})^\ell}K_{e^\omega}\,,\qquad A' = \sum_{e\in E(\mathcal{X})}[e^{\ell'}]_{e^\alpha}^{e^\omega}\,, \tag{4.1}$$

we obtain

$$(K_x^{-1}\mathfrak{F}_{x,y})_{x,y\in V(\mathcal{X})} = \frac{1}{1-A'} \tag{4.2}$$

and the proof is finished in the case that $\mathcal{X}$ is finite, because the matrix A' is precisely that obtained from A by substituting ℓ' for ℓ.

If $\mathcal{X}$ has infinitely many vertices, we approximate it, using Lemma 3.7, by finite graphs. Denote by $\mathfrak{F}^n_{\star,\dagger}(\ell)$ and $\mathfrak{G}^n_{\star,\dagger}(\ell')$ the enriched path series and path series respectively in $\mathcal{B}(\star,n)$, and write

$$K_\star\cdot\mathfrak{F}(\ell) = \lim_{n\to\infty}\mathfrak{F}^n_{\star,\dagger}(\ell) = \lim_{n\to\infty}\mathfrak{G}^n_{\star,\dagger}(\ell') = \mathfrak{G}(\ell')$$

to complete the proof.

5. Graphs and matrices

Graphs can be studied through their *adjacency* and *incidence* matrices. We give here the relevant definitions and obtain an extension of a theorem by Hyman Bass [Bas92] on the Ihara-Selberg zeta function. We will use power series with coefficients in a matrix ring, and fractional expressions in matrices; by convention, we understand 'X/Y' as '$X\cdot Y^{-1}$'.