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PROPOSITION 5.6 ([Ami80, Equation 4.4]). *Let X be a power series in t over a matrix ring, such that $X(0) = \mathbf{1}$. Then*

$$\det X = \exp\left(-\int \operatorname{tr}\left(\frac{\mathbf{1} - X}{Xt}\right) dt\right),$$

where the integration is the formal linear operation on power series that maps t^r to $t^{r+1}/(r+1)$.

We then have, using Lemmas 5.2 and 5.3,

$$\begin{aligned} \frac{\det M}{(1 + (1 - u)t)^n (1 - (1 - u)^2 t^2)^m} &= \det \frac{M}{\mathbf{1} + (1 - u)Jt} \\ &= \exp\left(-\int \operatorname{tr} \frac{\mathbf{1} + (1 - u)Jt - M}{Mt} dt\right) \\ &= \exp\left(-\int \text{series counting non-trivial circuits,} \right. \\ &\quad \left. \text{length shifted down by one} dt\right) \\ &= \exp\left(-\int \operatorname{tr} \frac{(1 - (1 - u)^2 t^2)\mathbf{1} - P}{Pt} dt\right) \\ &= \det \frac{P}{1 - (1 - u)^2 t^2} = \frac{\det P}{(1 - (1 - u)^2 t^2)^{|V(X)|}}. \end{aligned}$$

6. SECOND PROOF OF THEOREM 2.4

Let $P = [\star, \dagger]$ be the set of paths in \mathcal{X} from \star to \dagger . As we shall apply the principle of inclusion-exclusion [Wil90], it will be helpful to compute in $\Pi = \mathbf{Z}[[P]]$, the \mathbf{Z} -module of functions from the set of paths to \mathbf{Z} . We embed subsets of P in Π by mapping a subset to its characteristic function:

$$P \supset A \mapsto \chi_A, \quad \text{with } (\pi)\chi_A = \begin{cases} 1 & \text{if } \pi \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Let \mathcal{B} be the subset of bounded non-negative elements of Π (i.e. functions f such that there is a constant N with $0 \leq (\pi)f < N$ for all paths π). If ℓ is a complete labelling of \mathcal{X} , there is an induced labelling $\ell_*: \mathcal{B} \rightarrow \mathbf{k}$ given by

$$(f)\ell_* = \sum_{\pi \in P} (\pi)f \pi^\ell.$$

Note that the sum, although infinite, defines an element of \mathbf{k} due to the fact that ℓ is complete.

DEFINITION 6.1 (Bump Scheme). Let $e \in E(\mathcal{X})$ and $v \in V(\mathcal{X})$. A *squiggle along e* is a sequence $(e, \bar{e}, \dots, e, \bar{e})$. A *squiggle at v* is a squiggle along e for some edge e such that $e^\alpha = v$.

Let $\pi = (v_0, e_1, \dots, e_n, v_n)$ be a path of length n in \mathcal{X} . A *bump scheme* for π is a pair $B = ((\beta_0, \dots, \beta_n), (\gamma_1, \dots, \gamma_n))$, with

- for all $i \in \{0, \dots, n\}$, a finite (possibly empty) sequence $\beta_i = (\beta_{i,1}, \dots, \beta_{i,t_i})$ of squiggles at v_i ;
- for all $i \in \{1, \dots, n\}$, a squiggle γ_i along e_i .

The *weight* $|B|$ of the bump scheme B is defined as

$$|B| = \sum_{i=0}^n \sum_{j=1}^{t_i} (|\beta_{i,j}| - 1) + \sum_{i=1}^n |\gamma_i|.$$

Given a path π and a bump scheme $B = (\beta, \gamma)$ for π , we obtain a new path $\pi \vee B \in P$, by setting

$$\pi \vee B = \beta_{0,1} \cdot \dots \cdot \beta_{0,t_0} \gamma_1 e_1 \beta_{1,1} \cdot \dots \cdot \gamma_n e_n \beta_{n,1} \cdot \dots \cdot \beta_{n,t_n},$$

where the product denotes concatenation.

We now define a linear map $\phi: \Pi \rightarrow \Pi[[u]]$ by setting, for $f \in \Pi$ and $\pi \in P$,

$$(\pi)((f)\phi) = \sum_{(\rho, B): \rho \vee B = \pi} (u - 1)^{|B|} (\rho)f,$$

where the sum ranges over all pairs (ρ, B) where $\rho \in P$ and B is a bump scheme for ρ such that $\rho \vee B = \pi$. Note that the sum is finite because the edges of ρ and of B form subsets of those of π .

LEMMA 6.2. For any path π we have

$$(6.1) \quad (\pi)((\chi_P)\phi) = u^{\text{bc}(\pi)}.$$

Proof. Say $\pi = (\pi_1, \dots, \pi_n)$ has $m \geq 0$ bumps, at indices b_1, \dots, b_m so that $\pi_{b_i} = \overline{\pi_{b_{i+1}}}$. We will show that the evaluation at π of the left-hand side of (6.1) yields u^m .

We claim there is a bijection between the subsets C of $\{1, \dots, m\}$ and the pairs (ρ_C, B_C) where ρ_C is a path and B_C is a bump scheme for ρ_C with $\pi = \rho_C \vee B_C$; and further $|B_C| = |C|$.

First, take a ρ and a $B = (\beta, \gamma)$ such that $\rho \vee B = \pi$. The path $\rho \vee B$ is obtained by shuffling together the edges of ρ and B , and this partitions the

edges of π in two classes, namely (i) those coming from ρ and (ii) those coming from β and γ . Let $C \subset \{1, \dots, m\}$ be the indices of the bumps b_i in π coming from B , i.e. such that π_{b_i} and $\pi_{b_{i+1}}$ belong to the class (ii). One direction of the bijection is then $(\rho, B) \mapsto C$.

Conversely, given a subset C consider the set $D = \{b_i \mid i \in C\}$. Split it in maximal-length runs of consecutive integers $D = D_1 \sqcup \dots \sqcup D_t$. For all runs D_i do the following: to $D_i = \{j, j+1, \dots, j+2k-1\}$ of even cardinality associate a squiggle γ_j of length $2k$ along π_j ; to $D_i = \{j, j+1, \dots, j+2k-2\}$ of odd cardinality associate a squiggle $\beta_{j,l}$ of length $2k$ at v_{j-1} ; then delete in π the edges $\pi_j, \dots, \pi_{j+2k-1}$. This process constructs a bump scheme $B = (\beta, \gamma)$ while pruning edges of π , giving a path γ with $\gamma \vee B = \pi$. These two constructions are inverses, proving the claimed bijection.

It now follows that

$$(\pi)(\chi_P)\phi = \sum_{C \in \{1, \dots, m\}} (u-1)^{|B_C|} = \sum_{r=0}^m (u-1)^r \binom{m}{r} = u^m. \quad \square$$

Let $\ell' : E(\mathcal{X}) \rightarrow \mathbf{k}[[u]]$ be defined by

$$e^{\ell'} = \frac{1}{1 - (e\bar{e})^\ell (1-u)^2} e^\ell K_{e^\omega}.$$

We prove Theorem 2.4 by noting that $\mathfrak{G}(\ell) = (\chi_P)\ell_*$, that $\mathfrak{F}(\ell) = (\chi_P\phi)\ell_*$, and that for any $f \in \Pi$ we have $(f\phi)\ell_* = K_*(f)\ell'_*$. To prove this last equality, take a path $\pi = (\pi_1, \dots, \pi_n)$ on vertices v_0, \dots, v_n . Then

$$(\chi_{\{\pi\}}\phi)\ell_* = \sum_B (u-1)^{|B|} (\pi \vee B)^\ell,$$

where the sum ranges over all bump schemes for π , and

$$K_*\pi^{\ell'} = K_{v_0} \frac{1}{1 - (u-1)^2(\pi_1\bar{\pi}_1)^\ell} \pi_1^\ell K_{v_1} \cdots \frac{1}{1 - (u-1)^2(\pi_n\bar{\pi}_n)^\ell} \pi_n^\ell K_{v_n}.$$

It is clear these last two lines are equal; for the power series expansion of the K_{v_i} correspond to all the possible squiggle sequences β_i at v_i , and the power series expansion of the $1/(1 - (u-1)^2(\pi_i\bar{\pi}_i)^\ell)$ correspond to all possible squiggles γ_i along π_i .