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8.1 Π QUASI-FREE

Let S, T be finite sets, and $\bar{\cdot}$ an involution on S. Consider the two presentations

$$\Pi = \langle S \mid s\bar{s} = 1 \ \forall s \in S \rangle ,$$
$$\Pi = \langle S \cup T \mid s\bar{s} = 1 \ \forall s \in S; \ t = 1 \ \forall t \in T \rangle .$$

Let $\Xi < \Pi$ be any subgroup, and let F' and G' be the generating series related to the first presentation. Clearly $F' = F$, as both series count the same objects in Π (regardless of Π's presentation); while

$$G(t) = \frac{G'\left(\frac{t}{1-|T|t}\right)}{1-|T|t} .$$

Indeed any word $w = w_1 \dots w_n$ in $S \cup T$ defining an element of Ξ can be uniquely decomposed as $w = t_0 s_1 t_1 \dots s_m t_m$, where $s_i \in S$, t_i are words in T for all i, and $s_1 \dots s_n$ defines an element of Ξ; moreover all choices of $s_1 \dots s_n$ defining an element of Ξ and words t_i in T give a distinct word w. It then suffices to note that the generating series for any of the t_i is $1/(1-|T|t)$.

Putting everything together, we obtain:

PROPOSITION 8.1. *Let Π be as above, $\Xi < \Pi$ a subgroup. Then*

$$\frac{F(t)}{1-t^2} = \frac{G\left(\frac{t}{1+|T|t+(|S|-1)t^2}\right)}{1+|T|t+(|S|-1)t^2} .$$

8.2 $\Pi = \mathbf{PSL}_2(\mathbf{Z})$

Let

$$\Pi = \mathbf{PSL}_2(\mathbf{Z}) = \langle a, b \mid a^2, b^3 \rangle ,$$

and let $\Xi < \Pi$ be any subgroup. We take $S = \{a, b, b^{-1}\}$.

We suppose Ξ is torsion-free, i.e. contains no element of the form waw^{-1} or $wb^{\pm 1}w^{-1}$. Let $\mathcal{X}$ be the Schreier graph of $(\Pi, \{a, b, b^{-1}\})$ relative to Ξ, as defined in Subsection 3.1; it is a trivalent graph whose vertex set is $\Xi \backslash \Pi$. Its vertices can be grouped in triples $w^{\Delta} = \{w, wb, wb^{-1}\}$ connected in triangles. Let $\mathcal{F}$ be the graph obtained from $\mathcal{X}$ by identifying each triple to a vertex. Explicitly,