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## TEICHMÜLLER SPACE AND FUNDAMENTAL DOMAINS OF FUCHSIAN GROUPS

by Paul SCHMUTZ SCHALLER

### 1. INTRODUCTION

There are a number of ways to define the Teichmüller space of Riemann surfaces. In this paper I treat an approach which is less common than others. Let  $\Gamma$  be a Fuchsian group which uniformizes a closed Riemann surface of genus  $g$ . Then a fundamental domain for  $\Gamma$  is chosen in a canonical way, namely as a polygon with  $4g$  sides such that opposite sides are identified. The Teichmüller space  $T_g$  of closed Riemann surfaces of genus  $g$  is then constructed by varying these polygons.

This construction of  $T_g$  by polygons was first done by Coldewey and Zieschang in an annex in [17], see also [18]; the construction includes the proof that  $T_g$  is homeomorphic to  $\mathbf{R}^{6g-6}$ . In [2], Buser gave a different, however indirect proof. Here, I propose a new construction and a new proof which is, in my eyes, easier and more transparent than the original one of Coldewey and Zieschang.

The main idea is the following. Let  $P(g)$  be a canonical polygon of  $4g$  sides which is the fundamental domain of a Fuchsian group uniformizing a closed Riemann surfaces of genus  $g$  (the definition of  $P(g)$  will include some technical subtleties, to be discussed in Section 3). Then “triangulate”  $P(g)$  into  $4g - 4$  triangles and one quadrilateral  $S$ . This can be done in such a way that these triangles are determined by  $6g - 5$  positive real numbers (corresponding to the lengths of the sides of the triangles) with the condition that the different triangle inequalities hold. It turns out that these  $6g - 5$  lengths, *taken as homogeneous parameters*, provide a parametrization of the Teichmüller space  $T_g$ . Since the set of reals for which the different triangle

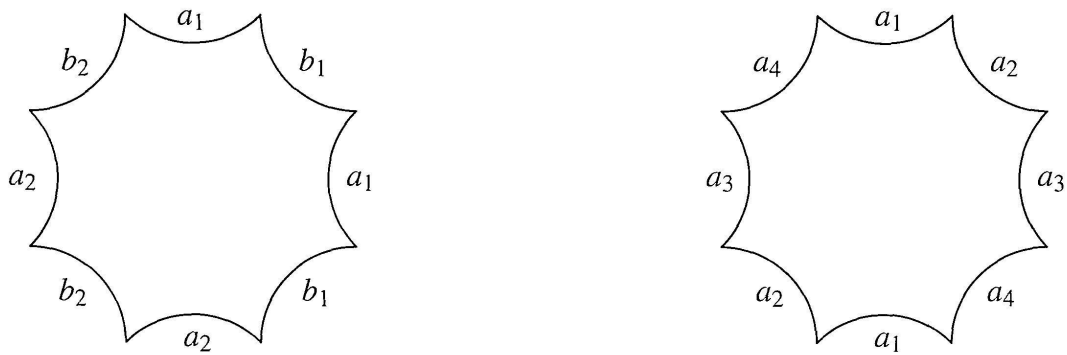


FIGURE 1

On the left hand side: usual identification

On the right hand side: identification chosen in this paper

inequalities hold is open and convex, this also proves that  $T_g$  is homeomorphic to  $\mathbf{R}^{6g-6}$ .

Let  $P$  be a polygon of  $4g$  sides which is the fundamental domain for a Fuchsian group  $\Gamma$  uniformizing a closed Riemann surface  $M$  of genus  $g$ . This means that we can write

$$M = \mathbf{H}/\Gamma$$

where  $\mathbf{H}$  is the upper halfplane. Usually,  $P$  is chosen such that the identification of the sides of  $P$  is that of the polygon on the left hand side in Figure 1. The construction described above would equally work for these polygons. For the following reasons I prefer to choose the identification (compare the polygon on the right hand side of Figure 1) such that opposite sides are identified. First the sides of  $P$  correspond to simple (this means with no selfintersections) closed curves in  $M$  and if opposite sides are identified, then these simple closed curves intersect transversally (which is not the case with the usual identification). Secondly, the vertices of  $P$  correspond to a (unique) point  $Q$  in  $M$ ; with the usual identification,  $Q$  is completely arbitrary while with the identification chosen here, there is a natural choice for  $Q$  in the case of hyperelliptic Riemann surfaces, namely, as a Weierstrass point. See Section 6 for details.

In this paper, I only treat the case of Fuchsian groups which uniformize closed Riemann surfaces. In a straightforward way, the construction and proof could be extended to all finitely generated Fuchsian groups. Note that concerning the original construction and proof in [17] (mentioned above) the corresponding generalization has been worked out by Coldewey in his thesis [3].

The paper is structured as follows. In Section 2 the basic definitions of hyperbolic geometry and Fuchsian groups are given. Section 3 defines the

canonical polygons. Section 4 provides the necessary material from hyperbolic trigonometry, it contains also some lemmas needed later. Section 5 contains the proof of the main theorem and Section 6 gives some applications, mainly concerning hyperelliptic Riemann surfaces. More precisely, I give a new proof of a geometric characterization of hyperelliptic Riemann surfaces which first appeared in [14] (I thank very much Feng Luo who, by his comments on [14], has contributed to the idea of this new proof). I also show (and this is a new result) that the Teichmüller space  $T_g$  for  $g = 2$  can be parametrized by 7 geodesic length functions, taken as homogeneous parameters. This is the optimum parametrization of Teichmüller space by geodesic length functions which one can expect.

I spoke about the content of this paper in lectures of the Troisième Cycle Romand de Mathématiques (Lausanne 1997); I thank the participants for their comments.

## 2. HYPERBOLIC GEOMETRY AND FUCHSIAN GROUPS

The material of this section and of parts of the following section is standard, see for example [1], [4], [5], [6], [7], [8], [15].

DEFINITION. (i)  $\mathbf{H} = \{z = (x, y) \in \mathbf{C} : y > 0\}$  denotes the *upper halfplane*. The *hyperbolic metric* on  $\mathbf{H}$  is given by

$$dz = \frac{1}{y}(dz)_E$$

where  $(dz)_E$  is the standard Euclidean metric on  $\mathbf{C}$  and  $y$  is the imaginary part of  $z$ .

(ii) Define

$$\mathrm{SL}(2, \mathbf{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = 1; a, b, c, d \in \mathbf{R} \right\}$$

and

$$\mathrm{PSL}(2, \mathbf{R}) = \mathrm{SL}(2, \mathbf{R})/\sim$$

with  $A \sim B$  if and only if  $A = \pm B$  for  $A, B \in \mathrm{SL}(2, \mathbf{R})$ . Let  $\gamma \in \mathrm{SL}(2, \mathbf{R})$ ,

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$