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A FREE GROUP ACTING ON $\mathbf{Z}^2$ WITHOUT FIXED POINTS

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ABSTRACT. The group of all orientation-preserving affine transformations of the plane has a non-abelian free subgroup which stabilizes $\mathbf{Z}^2$ and which acts on $\mathbf{Z}^2$ without non-trivial fixed points.

INTRODUCTION

Let G be a group acting on a non-empty set X. The following two conditions are known to be equivalent (see [D], and Theorems 4.5 and 4.8 in [W]):

(a) *there exists a non-abelian free subgroup of G whose action on X is locally commutative;*

(b) *there exists a G-paradoxical decomposition of X using 4 pieces, namely a partition of X in parts P_0, P_1, P_2, P_3 and elements α_0, α_1, α_2, α_3 in G such that*

$$X = P_0 \sqcup P_1 \sqcup P_2 \sqcup P_3 = \alpha_0(P_0) \sqcup \alpha_1(P_1) = \alpha_2(P_2) \sqcup \alpha_3(P_3).$$

Moreover, in the situation of (b), it can be shown that the subgroup of G generated by ${\alpha_0}^{-1}\alpha_1$ and ${\alpha_2}^{-1}\alpha_3$ is free of rank 2. (The symbol $\sqcup$ denotes disjoint union. Recall that an action of a group H on X is *locally commutative* if the stabilizer $\{h \in H \mid h(x) = x\}$ is commutative for all $x \in X$, i.e. if two elements of H which have a common fixed point commute; trivial examples of locally commutative actions are actions *without non-trivial fixed points*, for which $\{h \in H \mid h(x) = x\}$ is reduced to $\{1\}$ for all $x \in X$.)

For example, the group $SO_3(\mathbf{R})$ of rotations of the unit sphere $\mathbf{S}^2$ has such a free subgroup: this was discovered by F. Hausdorff (see, e.g., [Ś], or Theorem 2.1 in [W]). It implies the following result, for which we refer to [BT] and Theorem 3.11 in [W]; we denote by $SG_3(\mathbf{R})$ the group of all orientation-preserving isometries of $\mathbf{R}^3$.

THE BANACH-TARSKI PARADOX. *Any two bounded subsets U and V of the 3-dimensional Euclidean space $\mathbf{R}^3$ with non-empty interiors are* $SG_3(\mathbf{R})$*-equidecomposable. In other words, one can partition U into a finite number of pieces and reconstruct V from the same number of respectively* $SG_3(\mathbf{R})$*-congruent pieces.*

The Banach-Tarski paradox holds similarly for higher dimensional Euclidean spaces, but not for $\mathbf{R}$ and $\mathbf{R}^2$; the reason is that neither $SG_1(\mathbf{R})$ nor $SG_2(\mathbf{R})$, which are soluble groups, contain free subgroups of rank 2. (There are other known examples of free groups acting without non-trivial fixed points on familiar spaces. See e.g., [B], [DS], and [S2]. The proof of the Banach-Tarski paradox requires the axiom of choice, because the proof of the equivalence of conditions (a) and (b) requires it. But similar paradoxes hold for rational spheres of the form $(\sqrt{q}\,\mathbf{S}^2) \cap \mathbf{Q}^3$, as can be shown *without* the axiom of choice from the countability of rational spheres. See [S1], and [S3].) In dimension 2, von Neumann has exhibited a Banach-Tarski paradox with respect to the group $SA_2(\mathbf{R})$ of affine transformations of $\mathbf{R}^2$ that preserve area and orientation ([V], and Theorem 7.3 of [W]). The following problem was raised in [MW]; see also the discussion which follows Proposition 7.1 in [W].

PROBLEM ([MW], [W]). *Does* $SA_2(\mathbf{R})$ *contain a free subgroup of rank* 2 *whose action on* $\mathbf{R}^2$ *is locally commutative ?*

Indeed, these authors asked more specifically if the group generated by

$$\alpha\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$\beta\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

satisfies the requirements of the problem. We observe here that the answer is "no", because both $\alpha^{-2}\beta^2$ and $\alpha^{-1}\beta^{-1}\alpha\beta$ fix the origin.

Though we cannot solve the above problem, the purpose of this note is to show that, if one replaces $\mathbf{R}^2$ by $\mathbf{Z}^2$, the new problem has a positive solution. In fact, we will prove the following result, which shows somewhat more, namely that the action on $\mathbf{Z}^2$ may be an action without non-trivial fixed points, rather than only locally commutative. We denote by $\mathrm{SA}_2(\mathbf{Z})$ the group of all transformations $\vec{x} \mapsto A\vec{x} + \vec{a}$ of $\mathbf{Z}^2$, with $A \in \mathrm{SL}_2(\mathbf{Z})$ and $\vec{a} \in \mathbf{Z}^2$.

THEOREM. *The group* $\mathrm{SA}_2(\mathbf{Z})$ *has a free subgroup* F_2 *of rank* 2 *which acts on* $\mathbf{Z}^2$ *without non-trivial fixed points, namely the subgroup generated by*

$$\zeta: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 7 & 3 \\ 9 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and

$$\eta: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 94 & 39 \\ 147 & 61 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

The theorem implies the existence of a partition of $\mathbf{Z}^2$ into three pieces P, Q and R such that the six pieces P, Q, R, $P \sqcup Q$, $Q \sqcup R$, $R \sqcup P$ are pairwise F_2-congruent, without the axiom of choice ([S0], and Corollary 4.12 in [W]).

As observed in the discussion which follows Proposition 7.1 in [W], it is known that the above theorem does not carry over to $\mathbf{R}^2$; more precisely, it is known that a subgroup of $\mathrm{SA}_2(\mathbf{R})$ which acts on $\mathbf{R}^2$ without non-trivial fixed points is soluble, and consequently does not contain non-commutative free subgroups.

PROOF OF THE MAIN RESULT

Recall that a matrix in $\mathrm{SL}_2(\mathbf{Z})$ is *hyperbolic* if the absolute value of its trace is strictly larger than 2, or equivalently if it has an eigenvalue of absolute value strictly larger than 1.

LEMMA 0. *The subgroup of* $\mathrm{SL}_2(\mathbf{Z})$ *generated by*

$$\begin{pmatrix} 7 & 3 \\ 9 & 4 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 94 & 39 \\ 147 & 61 \end{pmatrix}$$

is free of rank 2 *and all its elements distinct from the identity are hyperbolic.*